The Method of Characteristics

Recall that the first order linear wave equation

$$u_t + cu_x = 0, \quad u(x,0) = f(x)$$

is constant in the direction $(1, c)$ in the (t, x)-plane, and is therefore constant on lines of the form $x - ct = x_0$. To determine the value of u at (x, t), we go backward along these lines until we get to $t = 0$, and then determine the value of u from the initial condition. The result is $u(x,t) = u(x - ct, 0) = f(x - ct)$.

There are many extensions to this simple idea. We begin by describing the situation for linear and nearly linear equations.

1 Homogeneous transport equations

We can carry out this same idea for equations of the form

$$u_t + c(x,t)u_x = 0, \quad u(x,0) = f(x), \quad -\infty < x < \infty. \tag{1}$$

Let $X(T)$ be any "trajectory" - think of it as a curve in the (x, t) plane, where (X, T) are supposed to be the (x, t) coordinates. How does u evolve as we move along this trajectory?

$$\frac{d}{dT}u(X(T), T) = X'(T)u_x(X(T), T) + u_t(X(T), T)$$

by the chain rule. If we happen to pick $X'(T) = c(X(T), T)$, then

$$\frac{d}{dT}u(X(T), T) = c(X(T), T)u_x(X(T), t) + u_t(X(T), T) = 0$$

by virtue of equation (1). Thus u is constant along ALL curves which are solutions of the ODE $X'(T) = c(X, T)$. To solve for u at some (x, t), we go "backward" along this curve until we hit time zero, and since u is constant along this curve, we find that the value of u is determined by the initial condition. In other words, if $X(t) = x$, then $u(x, t) = u(X(0), 0) = f(X(0))$.

The curves $X(T)$ that solve the ODE

$$X'(T) = c(X, T), \quad X(t) = x, \tag{2}$$

are called *characteristics*. For the purpose of finding characteristics, (x, t) are *fixed constants*, and it is X and T that vary along characteristics.

Example 1. Solve $u_t + xu_x = 0$ with initial condition $u(x, 0) = \cos(x)$.
Solution. A characteristic curve ending at (x, t) will solve

$$X'(T) = X(T), \quad X(t) = x,$$

whose solution is $X(T) = x \exp(T - t)$. Since u is constant along the characteristic,

$$u(x, t) = u(X(0), 0) = \cos(xe^{-t}).$$

Example 2. We want to solve

$$yu_x = xu_y, \quad u(0, y) = 2y^2 \text{ for } y > 0 \tag{3}$$

Solution. This equation can be written in the form (1) as

$$u_x - \frac{x}{y} u_y = 0,$$

treating x like the time variable. Let $Y(X)$ denote characteristic curves, which is a solution to

$$Y'(X) - \frac{X}{Y} = 0.$$

Separating variables $YdY = -XdX$ leads to $X^2 + Y^2 = C$; in other words, characteristics are closed curves encircling the origin. If an implicitly defined characteristic curve passes through (x, y), it is described by $X^2 + Y^2 = x^2 + y^2$. Since the solution is constant along this curve, setting $X = 0$ and using the side condition in (3) gives

$$u(x, y) = u(X, Y) = 2Y^2 = 2(x^2 + y^2).$$

Notice that if a boundary condition were imposed on the entire y-axis, then characteristic curves would intersect this boundary both at $(0, y)$ and $(0, -y)$. Unless $u(0, y) = u(0, -y)$, this problem would not have a solution.

1.1 Inhomogeneous transport equations

We can also solve equations of the form

$$u_t + c(x, t)u_x = g(u, x, t), \quad u(x, 0) = f(x), \quad -\infty < x < \infty. \tag{4}$$

The only difference between this and equation (1) is that u is not constant along characteristics, but evolves according to

$$\frac{d}{dt} u(X(t), t) = g(u, X(t), t). \tag{5}$$

In other words, if we let $U(T) = u(X(T), T)$ be the solution *restricted to a single characteristic*, it solves an initial value problem, namely

$$U'(T) = g(U, X(T), T), \quad U(0) = u(X(0), 0) = f(X(0)).$$

Thus, to find u at some point (x, t), we go *backwards* along the characteristic that ends at x until time zero, then solve the ODE (5) *forwards* until $T = t$.

1.2 The method of characteristics for linear problems

We can summarize ideas above as an algorithm:

1. **Find the characteristic terminating at (x, t):** Solve $X'(T) = c(X, T)$ with the "final" condition $X(t) = x$. Note that the solution for $X(T)$ will depend on x and t as parameters.

2. **Determine the solution along a characteristic:** Solve $U'(T) = g(U, X(T), T)$ subject to initial condition $U(0) = U(X(0), 0)$. Again the solution depends on x and t as parameters.

3. **Find the solution at the endpoint of the characteristic:** The solution of the PDE at (x, t) is simply $u(x, t) = U(t)$.

Here are a couple examples of how this is used.

Example 1. Solve

$$u_t + (x + t)u_x = t, \quad u(x, 0) = f(x).$$

Solution. Characteristic curves solve the ODE

$$X'(T) = X + T, \quad X(t) = x.$$

This equation has a particular solution, $X_p = -T - 1$; the general solution is therefore $X(T) = Ce^T - T - 1$. Using the condition $X(t) = x$, we find that

$$X(T) = e^{T-t}(x + t + 1) - T - 1.$$

Now we need to find how u changes along the characteristic. We solve

$$U'(T) = T, \quad U(0) = f(X(0)) = f(e^{-t}(x + t + 1) - 1).$$

whose solution by direct integration is

$$U(T) = f(e^{-t}(x + t + 1) - 1) + \frac{1}{2}T^2.$$

Finally, the solution at (x, t) is simply the value at the endpoint of the characteristic

$$u(x,t) = U(t) = f(e^{-t}(x+t+1)-1) + \frac{1}{2}t^2.$$

Example 2. Solve the nonlinear problem

$$u_t + 3u_x = -u^2, \quad u(x,0) = f(x).$$

Solution. In this case, characteristics solve $X'(T) = 2$ with $X(t) = x$, so that $X = 2(T-t) + x$. Along each characteristic, the solution evolves as $U'(T) = -U^2(T)$ with $U(0) = f(X(0)) = f(-2t+x)$. This is nonlinear, but we can solve it since it is just a separable ODE which can be written $dU/U^2 = -dT$, so that integration gives $1/U = T + C$. Using the initial condition, one gets $C = \frac{1}{f(x-2t)}$ and

$$U(T) = \frac{1}{T + \frac{1}{f(x-2t)}}.$$

The final solution is obtained by setting $u(x,t) = U(t)$.

Example 3. Suppose water flows over a landscape whose elevation is described by $h(x)$. A simple model for surface water flow says that the flow velocity is equal (in the right units) to $-h'(x)$. It follows that if $u(x,t)$ is the depth of water, then the flux of u is $J = -h'(x)u$. In the absence of sources u satisfies the conservation equation $u_t + (-h'(x)u)_x = 0$, which can be written in the form (4) as

$$u_t - h'(x)u_x = h''(x)u. \tag{6}$$

The term on the right accounts for the fact that water will accumulate in valleys where $h'' > 0$, and is depleted from hills where $h'' < 0$.

Consider a simple model for a valley where $h = x^2$, and suppose that the initial depth is localized as

$$u(x,0) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Since equation (6) reads $u_t - 2xu_x = 2u$, characteristics solve $X'(T) = -2X$ together with the terminal condition $X(t) = x$. The solution of this problem is

$$X(T) = xe^{2(t-T)}.$$

The solution on characteristics now solves $U' = 2U$ with initial condition

$$U(0) = \begin{cases} 1 & |X(0)| \leq 1 \\ 0 & |X(0)| > 1 \end{cases}$$

Therefore $U(T) = e^{2T}$ if $|X(0)| = |xe^{2t}| < 1$, or zero otherwise. It follows that when $t = T$,

$$u(x,t) = \begin{cases} e^{2t} & |x| \leq e^{-2t} \\ 0 & |x| > e^{-2t} \end{cases}$$

The depth of the fluid layer therefore increases exponentially, but its width decreases exponentially in a way such that $\int u dx$ remains constant.