

Dispersion relations, stability and linearization

1 Dispersion relations

Suppose that $u(x, t)$ is a function with domain $\{-\infty < x < \infty, t > 0\}$, and it satisfies a linear, constant coefficient partial differential equation such as the usual wave or diffusion equation. It happens that these type of equations have special solutions of the form

$$u(x, t) = \exp(ikx - i\omega t), \quad (1)$$

or equivalently,

$$u(x, t) = \exp(\sigma t + ikx). \quad (2)$$

We typically look for solutions of the first kind (1) when wave-like behavior which oscillates in time is expected, whereas (2) is used to investigate growth or decay in time. Plugging either (1) or (2) into the equation yields an algebraic relationship of the form $\omega = \omega(k)$ or $\sigma = \sigma(k)$, called the *dispersion relation*. It characterizes the dynamics of spatially oscillating modes of the form $\exp(ikx)$.

Here are a couple standard examples. For the wave equation $u_{tt} = c^2 u_{xx}$, we plug in a wave-like solution (1) to get $-\omega^2 \exp(ikx - i\omega t) = -c^2 k^2 \exp(ikx - i\omega t)$, or $\omega(k) = \pm ck$. This means there are traveling wave solutions of the form $u = \exp(ik(x \pm ct))$, which we might have guessed from the d'Alembert formula.

For the diffusion equation $u_t = D u_{xx}$, we use (2) to get $\sigma(k) = -Dk^2$. Since this is negative for all k , it is expected that solutions which are superpositions of (2) also decay. This is consistent with the fundamental solution representation for the diffusion equation.

1.1 Stability

For dispersion relations of the form $\sigma = \sigma(k)$ stemming from (2), the sign of the real part of σ indicates whether the solution will grow or decay in time. If the real part of $\sigma(k)$ is negative for all k values, then any superposition of solutions of the form $\exp(\sigma t + ikx)$ will also appear to decay. On the other hand, if the real part of $\sigma(k)$ is positive for some values of k , then over time some components of a superposition will grow exponentially. The former case is called *stable*, whereas the latter is *unstable*. If the maximum of the real part of σ is exactly zero, the situation is called *marginally stable*. It is more difficult to assess the long term behavior in this case.

1.2 Phase and group velocity of waves

For dispersion relations of the form $\omega(k)$, a solution of the form (1) can be written

$$u(x, t) = \exp\left(ik\left[x - \frac{\omega(k)}{k}t\right]\right), \quad (3)$$

which we notice are waves traveling at speed $\omega(k)/k$; this is known as the *phase velocity*. If the phase velocity is different for each k , a superposition of many different waves will appear to spread out or *disperse*.

Surprisingly, if the superposition contains only wavenumbers near a central wavenumber k_0 , the wave does not appear to move at the phase velocity $\sigma(k_0)/k_0$, but at a different speed. To see how this happens, consider an initial condition which is a superposition of many different modes

$$u(x, 0) = \int_{-\infty}^{\infty} A(k)e^{ikx} dk,$$

where we think of $A(k)$ as the amount of wavenumber k . If u solves an equation with a given dispersion relation $\omega = \omega(k)$ then the complete solution can be written

$$u(x, t) = \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} dk. \quad (4)$$

In general, the integral oscillates considerably as k is varied, and so one expects significant cancellation.

Suppose that $A(k)$ is concentrated about some wavenumber k_0 ; for example, $A(k) = \exp(-(k - k_0)^2/\epsilon)$ where ϵ is small. One might think that $A(k)$ could be replaced with a delta function $\delta(k - k_0)$ as a good approximation, and then one recovers $u \sim \exp(ik_0x - i\omega(k_0)t)$. This turns out to be too simple, since the “sideband” wavenumbers near k_0 play a role as t becomes large. Instead, we Taylor expand $\omega(k) \approx \omega(k_0) + \omega'(k_0)(k - k_0)$, and plug into (4), giving

$$u(x, t) \approx e^{it[\omega'(k_0)k_0 - \omega(k_0)]} \int_{-\infty}^{\infty} A(k) e^{ik(x - \omega'(k_0)t)} dk.$$

The first factor just oscillates in time, whereas the integral is a traveling wave of the form $f(x - \omega'(k_0)t)$. This means that the entire superposition appears to travel at the *group velocity* $\omega'(k_0)$, which is in general different than the phase velocity $\omega(k_0)/k_0$. Only in the case where the dispersion relation is linear $\omega = ck$ do the phase and group velocities coincide.

Take as an example the Schrödinger equation

$$iu_t + u_{xx} = 0.$$

Looking for solutions of the form $u = \exp(ikx - i\omega t)$, one gets

$$\exp(ikx - i\omega t)[i(-i\omega) - k^2] = 0, \quad \text{therefore } \omega = k^2.$$

The phase velocity is $\omega(k)/k = k$, so a superposition of waves will disperse. The group velocity is $\omega'(k) = 2k$, which is twice the phase velocity.

2 Steady state solutions

Suppose we have a (possibly nonlinear) PDE of the form

$$u_t = R(u, u_x, u_{xx}, \dots) \quad (5)$$

A *steady state* (or equilibrium) solution u_0 is one for which $(u_0)_t \equiv 0$, so that it solves

$$R(u_0, (u_0)_x, \dots) = 0. \quad (6)$$

In addition, (5) and (6) might also satisfy boundary conditions. Note that a steady state solution solves an ordinary differential equation, rather than a PDE.

Example. Often, the solution to (6) is just a constant $u = u_0$ in *space* as well as time. For the diffusion equation with Dirichlet-type boundary conditions

$$u_t = u_{xx}, \quad u(0, t) = 2 = u(1, t), \quad (7)$$

it is easy to see that $u(x, t) = 2$ is a solution which does not depend on time or the space variable. In general, however, equilibria may depend on x ; for example, for the diffusion equation with mixed boundary conditions

$$u_t = u_{xx}, \quad u(0, t) = 0, \quad u_x(1, t) = 1, \quad (8)$$

the equilibrium solution solves a two-point boundary value problem

$$(u_0)_{xx} = 0, \quad u_0(0) = 0, \quad (u_0)_x(1) = 1, \quad (9)$$

whose solution is easily obtained as $u_0 = x$.

Example. Fisher's equation is a nonlinear diffusion equation

$$u_t = u_{xx} + u(1 - u), \quad -\infty < x < \infty. \quad (10)$$

We can easily find two constant solutions $u(x, t) = u_0$. They solve $u_0(1 - u_0) = 0$ so that $u_0 = 0, 1$. This is one hallmark of nonlinear equations: they often possess numerous steady state solutions.

Example. A similar nonlinear diffusion equation is the Allen-Cahn equation

$$u_t = u_{xx} + 2u(1 - u^2), \quad -\infty < x < \infty. \quad (11)$$

Constant solutions solve $u_0(1 - u_0^2) = 0$ so that $u_0 = 0, \pm 1$. We can also find non-constant steady state solutions to (11) by imposing the conditions

$$\lim_{x \rightarrow -\infty} u(x) = -1, \quad \lim_{x \rightarrow \infty} u(x) = 1.$$

A steady solution $u(x, t) = u(x)$ solves

$$u_{xx} + 2u(1 - u^2) = 0.$$

This is a so-called Hamiltonian equation, and can be turned into a first order equation by multiplying by u_x and integrating. The result is

$$\int u_{xx}u_x + 2u(1 - u^2)u_x dx = \frac{1}{2}u_x^2 + u^2 - \frac{1}{2}u^4 + C = 0,$$

where we used the facts that $u_{xx}u_x = \frac{1}{2}(u_x^2)_x$ and $f'(u)u_x = f(u)_x$. Imposing the conditions at $x = \pm\infty$ gives $C = -1/2$. The first order equation can now be written

$$u_x = \sqrt{u^4 - 2u^2 + 1} = 1 - u^2,$$

which can be solved by separating variables

$$\frac{du}{1 - u^2} = dx, \quad \text{therefore} \quad \frac{1}{2} \ln \left| \frac{1 + u}{1 - u} \right| = x + c$$

so that finally

$$u(x) = \tanh(x + c).$$

Since c is arbitrary, there are an infinite number of steady state solutions which are just translations of one another.

Example. It was observed long ago that shallow water waves can appear as isolated pulses traveling at speed V . If one runs along at the same speed, the wave appears steady. This can be modeled by the Korteweg-de Vries (KdV) equation

$$u_t - Vu_x + 6uu_x + u_{xxx} = 0.$$

Steady solutions $u(x, t) = u(x)$ solve

$$-Vu_x + 6uu_x + u_{xxx} = 0.$$

If we suppose $\lim_{x \rightarrow \pm\infty} u(x) = 0$, this equation can be directly integrated to give

$$-Vu + 3u^2 + u_{xx} = 0.$$

This is also a Hamiltonian equation, so we turn it into a first order equation by the same trick as in the previous example, giving

$$\frac{1}{2}u_x^2 - \frac{V}{2}u^2 + u^3 = 0$$

where the constant of integration was zero using the far field conditions. This equation can now be solved by separation of variables, which gives

$$u(x) = \frac{V}{2} \operatorname{sech}^2 \left(\frac{\sqrt{V}}{2}(x + c) \right),$$

where again c allows for translation. These are the famous KdV solitons.

3 Linearization

It is frequently useful to approximate a nonlinear equation with a linear one, since we know a lot more about linear equations. If $u_0(x)$ is a steady state of an equation of the form (5), then we can look for solutions of the form

$$u(x, t) = u_0(x) + \epsilon w(x, t) \tag{12}$$

where ϵ presumed to be a small parameter. Plugging into (5) and keeping only the terms of order ϵ always gives us a linear, time dependent equation for w . This equation is called the *linearization* of (5) about u_0 .

The linearization can be used in a variety of ways. If it happens to admit a dispersion relation, this can tell us about stability or the dispersive nature of the wave components. Linearization also provides a basis for a wide variety of nonlinear techniques, which are beyond the scope of this discussion. It should be acknowledged that linearization has its limits: if $w(x, t)$ becomes large, then the justification for ignoring nonlinear terms proportional to higher powers of ϵ breaks down.

Consider again Fisher's equation (10), where we had previously found steady state solutions $u_0 = 0, 1$. Inserting the perturbed solution (12) into (10) for $u_0 = 0$, one gets

$$\epsilon w_t = \epsilon w_{xx} + \epsilon w - \epsilon^2 w^2.$$

Keeping only terms of order ϵ , we get the linearized Fisher equation

$$w_t = w_{xx} + w. \tag{13}$$

This is a diffusion equation with a linear source term. Substituting (2) into it, one gets a dispersion relation $\sigma = -k^2 + 1$. Since $\sigma > 0$ when $|k| < 1$, we might expect that an arbitrary initial condition for w has components that both grow and decay exponentially. We would therefore say that $u_0 = 0$ is *linearly unstable*.

If we linearize about $u_0 = 1$, we get a different result. Setting $u = 1 + \epsilon w$ gives

$$\epsilon w_t = \epsilon w_{xx} - \epsilon w - \epsilon^2 w^2,$$

so that the linearization is now

$$w_t = w_{xx} - w. \quad (14)$$

This equation has a dispersion relation $\sigma = -k^2 - 1$, which is always negative. Thus any initial condition of the form $u(x, 0) = 1 + \epsilon w(x, 0)$ in the original equation should evolve in time in a way that has w decay and $u \rightarrow 1$ as $t \rightarrow \infty$. We call this situation *linearly stable*.

3.1 Examples of linearization

Example. A model for flame-front propagation is the Kuramoto-Sivashinsky equation

$$u_t = -u_{xxxx} - u_{xx} + \frac{1}{2}u_x^2.$$

It is easy to see that $u_0 = 0$ is a steady state solution. Letting $u = u_0 + \epsilon w$, we get

$$\epsilon w_t = -\epsilon w_{xxxx} - \epsilon w_{xx} + \epsilon^2 \frac{1}{2}w_x^2.$$

Therefore the linearization is

$$w_t = -w_{xxxx} - w_{xx}.$$

Looking for a dispersion relation of the form $w = \exp(\sigma t + ikx)$, we get

$$\sigma(k) = -k^4 + k^2.$$

Since $\sigma > 0$ for $|k| < 1$, $u = 0$ is an unstable equilibrium. Any small initial condition containing wavenumbers in this range will therefore always grow. Of course, the solution will eventually get so large that the approximation leading to the linearization is not valid. At this point, more advanced techniques are needed to investigate the behavior.

Example. A thin liquid film of height $h(x, t)$ evolves according to the equation

$$h_t = (h^3[-h_{xx} + Ah^{-3}]_x)_x, \quad (15)$$

where A describes the role of intermolecular forces (e.g. van der Waals forces). Any constant solution $h(x, t) = h_0$ is a steady state. Setting $h(x, t) = h_0 + \epsilon w$, we can use the Taylor series approximations

$$(h_0 + \epsilon w)^3 = h_0^3 + \epsilon 3h_0^2 w + \mathcal{O}(\epsilon^2), \quad (h_0 + \epsilon w)^{-3} = h_0^{-3} - \epsilon 3h_0^{-4} w + \mathcal{O}(\epsilon^2),$$

where $\mathcal{O}(\epsilon^2)$ refers to terms which are smaller than a constant times ϵ^2 . Then plugging $h = h_0 + \epsilon w$ into equation (15), we get

$$\epsilon w_t = ((h_0^3 + \epsilon 3h_0^2 w)[- \epsilon w_{xx} + h_0^{-3} - \epsilon 3Ah_0^{-4} w]_x)_x + \mathcal{O}(\epsilon^2),$$

so that retaining the ϵ size terms,

$$w_t = h_0^3(-w_{xxxx} - 3Ah_0^{-4}w_{xx}).$$

The corresponding dispersion relation is found from $w = \exp(\sigma t + ikx)$, giving

$$\sigma(k) = h_0^3(-k^4 + 3Ah_0^{-4}k^2),$$

which has a band of unstable wavenumbers $|k| < h_0^{-2}\sqrt{3A}$, provided $A > 0$. On the other hand, if $A < 0$, the film is stable.

Example. The so-called sine-Gordon equation is

$$u_{tt} = c^2u_{xx} - \sin(u).$$

If we set $u = \epsilon w(x, t)$, we use $\sin(\epsilon w) = \epsilon w + \mathcal{O}(\epsilon^2)$ which produces the linearized equation

$$w_{tt} = c^2w_{xx} - w.$$

This is a wave type equation, so it makes sense to use the form $w(x, t) = \exp(ikx - i\omega t)$ for the dispersion relation. Plugging in we get

$$-\omega^2 = -c^2k^2 - 1, \quad \omega(k) = \pm\sqrt{1 + c^2k^2}.$$

Since the phase velocity $\pm\sqrt{1 + c^2k^2}/k$ is not constant, waves disperse. On the other hand, they do not grow or decay in time.