First and second order linear wave equations

1 Simple first order equations

Perhaps the simplest of all partial differential equations is

$$u_t + cu_x = 0, \quad -\infty < x < \infty. \tag{1}$$

There are no boundary conditions required here, although to find a unique solution some kind of side condition is required. Equation (1) is sometimes called the transport equation, because it is the conservation law with the flux cu, where c is the transport velocity.

We can view (1) as the directional derivative of u in the direction $\mathbf{v} = (c, 1)$ where $\mathbf{v}$ is a vector in (x, t)-space. Therefore (1) means that the function $u(x, t)$ is **constant** on each line parallel to $\mathbf{v}$. These lines have the equations $x - ct = x_0$ for any constant x_0, and are known as *characteristic curves*, or simply characteristics. Since u is constant along such lines, it must be a function of $x - ct$ alone, and the most general solution of (1) is therefore

$$u(x, t) = f(x - ct). \tag{2}$$

If we were supplied with an initial condition to (1), we immediately find that $f(x) = u(x, 0)$. The solution (2) therefore merely translates the initial data at speed c as time progresses; such a solution is called a *travelling wave*. Note that the solution (2) can be obtained by other means, including Fourier transforms.

More generally, equations of the form

$$au_x + bu_y = 0 \tag{3}$$

have the same structure as (1). In this case the characteristic curves are of the form $bx - ay = C$ for any constant $C \in \mathbb{R}$. It follows that all solutions in this case have the form

$$u(x, t) = f(bx - ay) \tag{4}$$

for some arbitrary function $f()$. Not every possible set of boundary conditions for (3) are compatible. This happens since $u(x, y)$ is constant along characteristic curves, and the boundary data must be the same for boundary points lying on the same characteristic.

Example. Suppose we wanted to solve

$$u_x = u_y, \quad u(x, y) : \mathbb{R}^2 \to \mathbb{R}, \quad u(x, y) = \sin(x) \text{ on the line } y = x.$$

By (4) the most general solution is $u(x, y) = f(x + y)$. On the line $y = x$ we have $\sin(x) = u(x, y) = f(2x)$. This is a type of *functional equation*, whereby an unknown function $f()$ is given implicitly. Setting $z = 2x$, we have $x = z/2$ and $f(z) = \sin(z/2)$. The complete solution is therefore $u(x, y) = \sin((x + y)/2)$.

2 Second order wave equations

Now consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{5}$$

on the entire real line $x \in (-\infty, \infty)$. We can factor the linear operator to give

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Setting $v = u_t + cu_x$, we get two first order equations

$$\begin{aligned} v_t - cv_x &= 0, \\ u_t + cu_x &= v. \end{aligned}$$

(this is like solving linear systems by factoring as $ABx = b$: we can introduce an intermediate quantity $w = Bx$, solve $Aw = b$ and finally solve $w = Bx$). The general solution to the first equation is just $v = h(x + ct)$ for some function h. Now we must solve

$$u_t + cu_x = h(x + ct). \tag{6}$$

This is an *inhomogeneous* equation, and we can attempt to solve it by looking for solutions of the form $u = u_{\text{hom}} + u_p$ where u_p is a particular solution and u_{hom} solves the homogeneous equation

$$u_t + cu_x = 0. \tag{7}$$

The general solution to (7) is $u_{\text{hom}} = g(x - ct)$. We guess a particular solution of the form $u_p = f(x + ct)$. Plugging it gives $f'(s) = h(s)/2c$, which means that in principle $f(s)$ can be found by integration. Therefore the whole solution $u = u_{\text{hom}} + u_p$ must have the general form

$$u = g(x - ct) + f(x + ct). \tag{8}$$

There are two sets of characteristic lines, one going forward at speed c and the other going backward at speed c, along which the solution components $g()$ and $f()$ are constant, respectively.

Now we would like to satisfy the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x)$$

Inserting into the general solution (8) gives

$$f(x) + g(x) = u_0(x), \tag{9}$$

$$f'(x) - g'(x) = \tfrac{1}{c} v_0(x). \tag{10}$$

Integrating the second of these gives

$$f(x) - g(x) = \frac{1}{c}\int_0^x v_0(x')dx' + K \tag{11}$$

where K is some constant of integration (the lower bound on the integral was specified arbitrarily). We can now add and subtract (9) and (11) in order to find f and g:

$$\begin{aligned} f(x) &= \tfrac{1}{2}\left(u_0(x) + \tfrac{1}{c}\textstyle\int_0^x v_0(x')dx' + K\right), \\ g(x) &= \tfrac{1}{2}\left(u_0(x) - \tfrac{1}{c}\textstyle\int_0^x v_0(x')dx' - K\right) \end{aligned}$$

Therefore the complete solution to the initial value problem is (notice that K drops out)

$$u = f(x + ct) + g(x - ct) = \frac{1}{2}\left(u_0(x + ct) + u_0(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x')dx'. \tag{12}$$

This is known as d'Alembert's solution to the wave equation.

2.1 Other second order wave equations

The above method can be generalized to any second order PDE which can be factored as two transport equations. For example,

$$u_{xx} + (a-b)u_{xy} - ab u_{yy} = 0 \tag{13}$$

can be factored as

$$\left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right) u = 0.$$

which can be written as the system

$$\begin{aligned} v_x + a v_y &= 0, \\ u_x - b u_y &= v. \end{aligned}$$

Following the same logic as above, we see that the most general solution is

$$u(x,y) = f(y - ax) + g(y + bx)$$

for arbitrary functions f, g. In this case, the solution components each have their own "characteristic speed".

Example. We want a d'Alembert type solution for $u_{xx} + u_{xy} - 20u_{yy} = 0$ subject to initial conditions $u(x,0) = \phi(x)$ and $u_y(x,0) = \psi(x)$. Factoring gives

$$\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right) u = 0.$$

so that by the above discussion, a general solution can be written

$$u(x,y) = g(4x + y) + f(5x - y).$$

To satisfy the initial data, we need

$$g(4x) + f(5x) = \phi(x), \tag{14}$$

$$g'(4x) - f'(5x) = \psi(x). \tag{15}$$

Integration of the second equation gives

$$\frac{1}{4}g(4x) - \frac{1}{5}f(5x) = \int_0^x \psi(x')dx' + C. \tag{16}$$

Using elimination to solve the linear equations (14) and (16) gives

$$\begin{aligned} g(4x) &= \frac{4}{9}\phi(x) + \frac{20}{9}\int_0^x \psi(x')dx' + \frac{20C}{9}, \\ f(5x) &= \frac{5}{9}\phi(x) - \frac{20}{9}\int_0^x \psi(x')dx' - \frac{20C}{9}. \end{aligned}$$

It follows that

$$\begin{aligned} u(x,y) &= g(4x+y) + f(5x-y) = g(4(x+y/4)) + f(5(x-y/5)) \\ &= \frac{4}{9}\phi(x+y/4) + \frac{4}{9}\phi(x-y/5) + \frac{20}{9}\int_{x-y/5}^{x+y/4} \psi(x')dx'. \end{aligned}$$

2.2 Wave speed and domain of dependence

The formula (12) which solves (5) reveals that the solution at (x, t) only depends on the initial condition on the interval $I_d(x, t) = (x - ct, x + ct)$. This is because waves in the second order wave equation travel both left and right with speed c, but no faster. As a consequence, initial data outside the interval I_d cannot affect the solution at (x, t) since it cannot travel fast enough along characteristic lines. The interval I_d is known as the *domain of dependence*.

In contrast, consider the solution to the diffusion equation on a line

$$u(x, t) = \int_{-\infty}^{\infty} \frac{f(x_0)}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)} dx_0.$$

If the initial condition $f(x)$ is changed, the entire solution for $(x, t) \in \mathbb{R} \times \{t > 0\}$ will change, since the fundamental solution is never zero. The domain of dependence in this case is therefore the real line $I_d = \mathbb{R}$.