Higher dimensional PDEs and multidimensional eigenvalue problems

1 Problems with three independent variables

Consider the prototypical equations

$$\begin{aligned} u_t &= \Delta u \quad (Diffusion) \\ u_{tt} &= \Delta u \quad (Wave) \\ -u_{zz} &= \Delta u \quad (Laplace) \end{aligned}$$

where $\Delta u = u_{xx} + u_{yy}$ (recall there are no t derivatives in the Laplacian). The domain for $u(x, y, \cdot)$ will be a generalized cylinder where $(x, y) \in D$ and $t > 0$ or $z \in (a, b)$. For now, the planar region D is taken to be any bounded, open domain. We handle all three cases together because one variable can be separated out, leading to exactly the same eigenvalue problem in the x, y-variables.

While a variety of homogeneous boundary conditions on ∂D are possible, we will only consider the two most common,

$$u(x, y, \cdot) = 0, \quad (x, y) \in \partial D \quad (Dirichlet) \tag{1}$$

$$\nabla u(x, y, \cdot) \cdot \hat{\mathbf{n}} = 0, \quad (x, y) \in \partial D \quad (Neumann) \tag{2}$$

where $\hat{\mathbf{n}}$ is the normal vector to the boundary ∂D.

Let's look for solutions of the form $u = T(t)v(x, y)$ (for the diffusion or wave equation) or $u = Z(z)v(x, y)$ (for Laplace's equation). After separating variables, we get

$$\begin{aligned} \frac{T'}{T} &= \frac{\Delta v}{v} = -\lambda \quad (Diffusion) \\ \frac{T''}{T} &= \frac{\Delta v}{v} = -\lambda \quad (Wave) \\ \frac{-Z''}{Z} &= \frac{\Delta v}{v} = -\lambda \quad (Laplace) \end{aligned}$$

Therefore the resulting eigenvalue problem for v is the same in each case,

$$\Delta v + \lambda v = 0, \quad \text{plus boundary conditions (1) or (2).} \tag{3}$$

For the time being, suppose that we already know the eigenfunctions $v_n(x, y)$ and corresponding eigenvalues λ_n, $n = 1, 2, 3, \ldots$ (in practice, it is often be easier to enumerate these using two indices). We will show that with either of the above boundary conditions, Δ is a self adjoint operator. As a consequence, the eigenvalues are real and the eigenfunctions are orthogonal with respect to the L^2 inner product

$$\langle u, v \rangle = \int_D u(\mathbf{x})v(\mathbf{x})\, d\mathbf{x}. \tag{4}$$

We will also show that the eigenvalues happen to be non-negative.

The ODEs implied by separation of variables for T and Z and their solutions are the same as in the simpler, two variable case:

$$\begin{aligned} &T' = -\lambda T, \quad T = \exp(-\lambda t) \quad (Diffusion) \\ &T'' = -\lambda T, \quad T = \sin(\sqrt{\lambda} t), \cos(\sqrt{\lambda} t) \quad (Wave) \\ &Z'' = \lambda Z, \quad Z = \exp(\pm\sqrt{\lambda} t) \quad (Laplace). \end{aligned}$$

Taking a superposition of all possible separated solutions, the most general solutions are

$$\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{\infty} A_n \exp(-\lambda_n t) v_n(x,y) \quad (Diffusion) \\
u(x,y,t) &= \sum_{n=1}^{\infty} [A_n \cos(\sqrt{\lambda}_n t) + B_n \sin(\sqrt{\lambda}_n t)] v_n(x,y) \quad (Wave) \\
u(x,y,z) &= \sum_{n=1}^{\infty} [A_n \exp(\sqrt{\lambda}_n z) + B_n \exp(-\sqrt{\lambda}_n z)] v_n(x,y) \quad (Laplace)
\end{aligned}$$

The main issue, therefore, is to solve the eigenvalue problem. This is in general impossible by hand unless the domain of interest has some symmetry. Our examples include three cases, where D is a rectangle, a disk, and the surface of a sphere.

2 Self adjointness and properties of the Laplacian

The Laplacian turns out to be self-adjoint regardless of the domain we are working with, provided the boundary conditions are suitable. To find the adjoint, we first need the equivalent of integration by parts in higher dimensions.

2.1 Green's identity

Let $u, v : D \to \mathbb{R}$ be smooth functions. Applying the divergence theorem to the vector field $u\nabla v$,

$$\int_D \nabla\cdot(u\nabla v)\, d\mathbf{x} = \int_{\partial D} u\nabla v \cdot \hat{\mathbf{n}}\, d\mathbf{x}. \tag{5}$$

Using the "multidimensional product rule" $\nabla\cdot(u\nabla v) = \nabla u \cdot \nabla v + u\Delta v$, in (5), one obtains

$$\int_D u\Delta v\, d\mathbf{x} = -\int_D \nabla u \cdot \nabla v\, d\mathbf{x} + \int_{\partial D} u\nabla v \cdot \hat{\mathbf{n}}\, d\mathbf{x}. \tag{6}$$

This is known as *Green's identity*, or sometimes just Green's theorem. Note this result is just like integration by parts for definite integrals, where derivatives are moved around inside an integral, while paying the price of a boundary term.

2.2 Self adjointness

We can use (6) to compute the adjoint of Δ with respect to the inner product given by (4). The Laplacian is regarded as a differential operator acting on smooth functions (for example $C^\infty(D)$) which have either type of boundary condition (1) or (2). It follows by using Green's identity twice that

$$\begin{aligned}
\langle \Delta u, v\rangle = \int_D v\Delta u\, d\mathbf{x} = -\int_D \nabla v \cdot \nabla u\, d\mathbf{x} + \int_{\partial D} v\nabla u \cdot \hat{\mathbf{n}}\, d\mathbf{x} = \\
= \int_D u\Delta v\, d\mathbf{x} - \int_{\partial D} u\nabla v \cdot \hat{\mathbf{n}}\, d\mathbf{x} = \langle u, \Delta v\rangle.
\end{aligned} \tag{7}$$

The integrals on the boundary ∂D vanish because of the boundary conditions. The Laplacian is therefore self-adjoint with either Dirichlet or Neumann boundary conditions, just like its one dimensional counterpart ∂_{xx}.

2.3 Non-negativity of the eigenvalues

It is often possible to assess quantitative properties of eigenvalue problems without actually solving them. One way to do this is with the *Rayleigh quotient*, which is formed by taking the inner product of the eigenvalue equation $\mathcal{L}v + \lambda v = 0$ with v, resulting in a formula

$$\lambda = -\frac{\langle \mathcal{L}v, v\rangle}{\langle v, v\rangle}. \tag{8}$$

The expression on the right depends on the unknown eigenfunction v, so it's hard to imagine how this might be useful. If we specialize to our situation, however, we have

$$\lambda = -\frac{\int_D v\Delta v \, d\mathbf{x}}{\int_D v^2 \, d\mathbf{x}} = \frac{\int_D |\nabla v|^2 \, d\mathbf{x}}{\int_D v^2 \, d\mathbf{x}},$$

where the Green's identity was used. The expression on the right is certainly non-negative, regardless of what v might be. Notice that if $\lambda = 0$, then it must be that $\nabla v = 0$, so that the eigenfunction is a constant. This can only happen in the case of the Neumann boundary condition (2); for the Dirichlet boundary condition (1), eigenvalues are strictly positive.

There are other uses of Rayleigh quotient (8). For example, the minimum value this expression can take for *any* function $v \neq 0$ (not just eigenfunctions) is equal to the smallest eigenvalue. Minimization of this expression is the basis for the *Rayleigh-Ritz* algorithm for computation of eigenvalue problems.

3 Eigenfunctions on the rectangle

We now specialize to the region $D = \{0 < x < \pi, 0 < y < \pi\}$. The natural idea to solve the eigenvalue problem is (of course!) separation of variables. Letting $v(x, y) = X(x)Y(y)$, plugging into (3) and separating gives

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda \tag{9}$$

It follows that each term on the left must be a constant, since the right hand side does not depend on either x or y. Letting those constants be $-\lambda_x, -\lambda_y$, respectively, one gets

$$X'' + \lambda_x X = 0, \quad Y'' + \lambda_y Y = 0, \quad \lambda = \lambda_x + \lambda_y. \tag{10}$$

Suppose that Dirichlet boundary conditions on ∂D were imposed. This leads to the boundary conditions

$$X(0) = 0 = X(\pi), \quad Y(0) = 0 = Y(\pi).$$

The eigenvalue problems for X and Y are familiar, leading to eigenfunctions $X = \sin(nx)$ and $Y = \sin(my)$ for any $n, m = 1, 2, 3, \ldots$. It follows that solutions to the multidimensional eigenvalue problem can be enumerated using a double index

$$v_{nm}(x, y) = \sin(nx)\sin(my), \quad \lambda_{nm} = n^2 + m^2, \quad n, m = 1, 2, 3, \ldots.$$

A superposition of these functions by themselves give one version of a multidimensional Fourier series. Since the underlying operator is self-adjoint, orthogonality holds:

$$\int_D v_{nm} v_{n'm'} dx dy = 0 \quad \text{unless } n = n' \text{ and } m = m'.$$

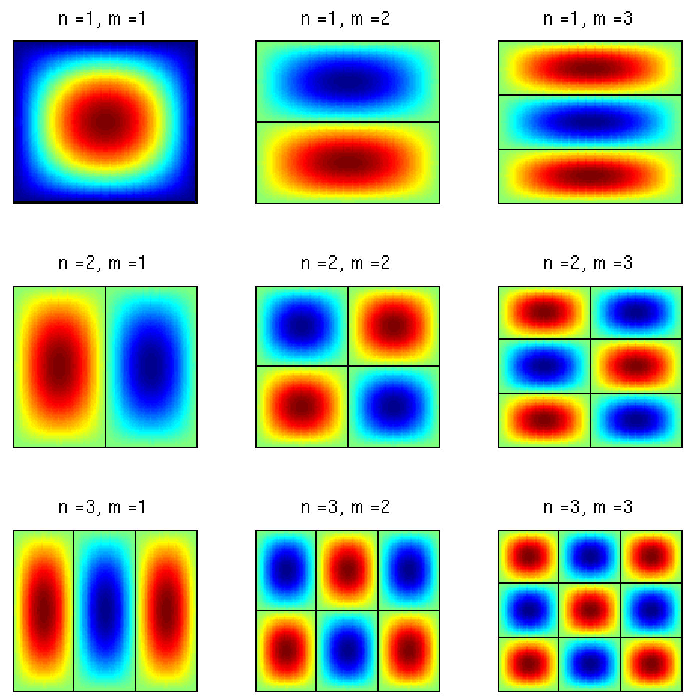

Figure 1: Nodal sets (level curves where $v = 0$) for the first several eigenfunctions on a square domain with Dirichlet boundary conditions.

It is not too hard to extend this example to other boundary conditions and higher dimensions. For example, with the Neumann boundary condition $\nabla v \cdot \hat{\mathbf{n}} = 0$, the boundary conditions for X, Y become

$$X'(0) = 0 = X'(\pi), \quad Y'(0) = 0 = Y'(\pi).$$

We have solved the corresponding eigenvalue problems previously; the solutions are $\lambda_x = n^2$ for $n = 0, 1, 2, 3, \ldots$ and $X = \cos(nx)$, with similar results for Y. Thus the two-dimensional eigenfunctions are

$$v_{nm}(x, y) = \cos(nx)\cos(my), \quad \lambda_{nm} = n^2 + m^2, \quad n, m = 0, 1, 2, 3, \ldots.$$

Notice that for this boundary condition, we get a constant eigenfunction corresponding to $\lambda = 0$.

To provide some intuition about eigenfunctions, it is often useful to visualize them in various ways. A common way is to graph the *nodes*, which are the level curves where $v(x, y) = 0$. The nodes for the first several eigenfunctions on the square domain are given in figure (1). As a general rule, as the eigenvalue increases, the nodal sets subdivide D into more and more regions.

4 Eigenfunctions on the disk

Now let $D = \{0 < r < a,\, 0 < \theta < 2\pi\}$. In polar coordinates, (3) is

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = -\lambda v.$$

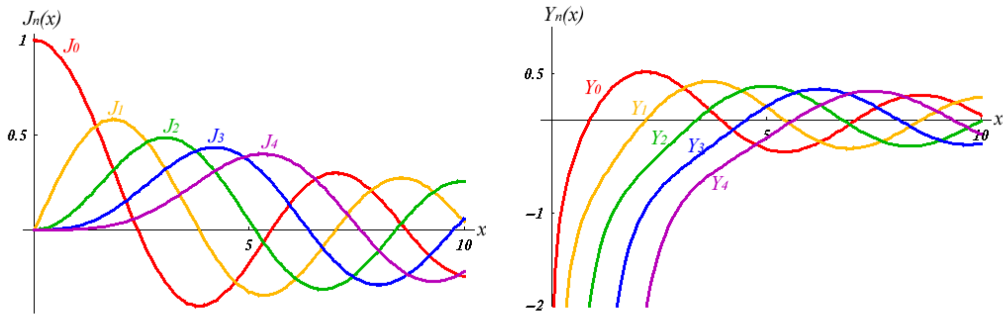

Figure 2: Graphs of some of the Bessel functions.

Looking for solutions of the form $v = R(r)\Theta(\theta)$, separating variables gives

$$\frac{r^2R'' + rR' + \lambda r^2 R}{R} + \frac{\Theta''}{\Theta} = 0. \tag{11}$$

Again each term must be a constant; in particular we can set $\Theta''/\Theta = -\lambda_\theta$. The boundary conditions for Θ are periodic, so the solutions are immediately

$$\Theta = \cos(n\theta), \sin(n\theta), \quad \lambda_\theta = n^2, \quad n = 0, 1, 2, \ldots$$

The equation in R gives something new. Setting $\Theta''/\Theta = -\lambda_\theta = -n^2$ in (11), we get

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0 \tag{12}$$

In the case $\lambda = 0$, we get Euler's equation (recall the general solution is $R(r) = Ar^n + Br^{-n}$ or $R(r) = A + B\ln n$ if $n = 0$). For the Dirichlet boundary condition, none of these solutions are admissible, whereas for the Neumann boundary condition, we can have constant-valued solutions.

If $\lambda > 0$, we can simplify the problem a little by a change of variables $\rho = \sqrt{\lambda}r$. Thus $R'(r) = \sqrt{\lambda}R'(\rho)$ and $R''(r) = \lambda R''(\rho)$, and $R(\rho)$ solves

$$\rho^2R'' + \rho R' + (\rho^2 - n^2)R = 0. \tag{13}$$

This is known as *Bessel's equation* of order n. Notice that the parameter λ has been eliminated; it will re-appear when the boundary conditions are invoked.

Unfortunately, (13) does not possess closed form solutions. To get an idea of what solutions look like, let's approximate near $\rho = 0$. We can't eliminate the first two terms in (13) since we have no idea if the smallness of ρ is counteracted by the possibility that R'' and R' are large. On the other hand, $\rho^2 R$ is smaller than the other terms, allowing us to ignore it. The approximate equation is then $\rho^2R'' + \rho R' - n^2R = 0$, which is just the Euler equation. Therefore for each n we expect two linearly independent solutions we call $J_n(\rho)$ and $Y_n(\rho)$ having the behavior

$$J_n(\rho) \sim \begin{cases} \rho^n & n > 0 \\ 1 & n = 0, \end{cases} \qquad Y_n(\rho) \sim \begin{cases} \rho^{-n} & n > 0 \\ \ln\rho & n = 0, \end{cases} \tag{14}$$

as $\rho \to 0$ (note n does not index eigenvalues, but is simply the order of the Bessel equation). These solutions are called *Bessel functions*; they are depicted graphically in figure 2.

The eigenvalues, of course, depend on boundary conditions. Take, for example, the Dirichlet condition $v(a, \theta) = 0$, which means $R(r = a) = 0$. Since the Bessel functions Y_n are singular at the origin, they can be ignored for now. We need to know where the Bessel functions J_n have zeros. There is no simple formula for these; we deal with this issue by simply inventing notation

$$\beta_{nm} = m\text{th positive zero of } J_n(\rho).$$

(note that Haberman's text uses z_{mn} instead). It turns out that each Bessel function J_n has an infinite number of zeros, so $m = 1, 2, 3, \ldots$. (in fact, it is helpful to think of Bessel functions as distorted versions of $\cos(\rho)$). It follows that the eigenvalues λ are determined by $R(r = a) = J_n(\sqrt{\lambda}a) = 0$. This means that for each n, $\sqrt{\lambda}a$ must be a zero of J_n, and the eigenvalues are

$$\lambda_{nm} = \left(\frac{\beta_{nm}}{a}\right)^2, \quad 0 \leq n < \infty, \qquad 1 \leq m < \infty.$$

The corresponding eigenfunctions in original variables are

$$v(r,\theta) = \begin{cases} J_0(\beta_{0m}r/a) & n = 0 \\ J_n(\beta_{nm}r/a)\cos(n\theta),\ J_n(\beta_{nm}r/a)\sin(n\theta), & n > 0. \end{cases}$$

Other eigenvalue problems have similar results. For example, in the case of Neumann boundary conditions, one needs to determine the zeros of $J'(\rho)$.

4.1 No really, what *are* the Bessel functions?

Unfortunately, many solutions to differential equations simply cannot be written in terms of elementary functions. An alternative is to use power series; in our case we look for solutions of Bessel's equation of the form

$$R = \rho^\alpha \sum_{k=0}^{\infty} a_k \rho^k, \quad a_0 \neq 0,$$

where α and a_n are to be determined. Plugging into equation (13) and re-indexing the sum on the last term gives

$$\rho^\alpha \left\{ \sum_{k=0}^{\infty} [(\alpha+k)(\alpha+k-1) + (\alpha+k) - n^2] a_k \rho^{k-2} + \sum_{k=2}^{\infty} a_{k-2}\rho^{k-2} \right\} = 0.$$

Each coefficient of ρ^{k-2} must equal zero for this to be true. Thus for $k = 0$ or $k = 1$,

$$[\alpha^2 - n^2]a_0 = 0, \quad [(\alpha+1)^2 - n^2]a_1 = 0.$$

The first equation implies that α is either n or $-n$; this is consistent with the approximations in (14). The second equation implies $a_1 = 0$, which will lead to the conclusion that all odd coefficients are zero.

In general, the coefficients can be found recursively by solving

$$a_k = -\frac{a_{k-2}}{(\alpha+k)^2 - n^2}, \quad k = 2, 3, 4, \ldots$$

Once these are found, one obtains

$$J_n(\rho) = \sum_{k=0}^{\infty} (-1)^k \frac{(\rho/2)^{n+2k}}{k!(n+k)!}$$

where the conventional choice $a_0 = 2^{-n}/n!$ was made. By the elementary ratio test, the series converges for all ρ.

This is still not a very satisfactory situation, since the power series is hard to interpret. In practice, there are several other ways to quantify Bessel functions, including integral representations, approximations, and computational packages like MATLAB. There are also a million identities concerning Bessel functions and their derivatives.

4.2 Orthogonality

Notice that Bessel functions $J_n(\beta_{nm} r/a)$ are themselves eigenfunctions of a differential operator, namely

$$\mathcal{L} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \tag{15}$$

Its not hard to show that this operator is self-adjoint with respect to the inner product

$$\langle u, v \rangle = \int_0^a u(r)v(r) r dr.$$

Notice the weight r in the integrand, which should come as no surprise, since the operator (15) arises from the use of polar coordinates. It follows that the scaled Bessel functions are orthogonal:

$$\langle J_n(\beta_{nm} r/a), J_n(\beta_{nk} r/a) \rangle = 0, \quad m \neq k.$$

(note that n needs to be the same since different n implies different operators!).

5 Spherical harmonics

In spherical coordinates (r, ϕ, θ) (see figure 3), Laplace's equation is

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[\frac{u_{\phi\phi}}{\sin^2(\theta)} + \frac{1}{\sin\theta}(\sin\theta\, u_\theta)_\theta\right] = 0. \tag{16}$$

Here $0 \leq \phi < 2\pi$ is the longitudinal variable, whereas $0 \leq \theta < \pi$ measures the latitude where $\theta = 0$ is the north pole (note that in some texts, this notation is opposite). Suppose we want solutions to $\Delta u = 0$ of the form $u = R(r)v(\phi, \theta)$. After separating out the r variable, we are left with an eigenvalue problem for $v(\phi, \theta)$

$$\Delta_s v + \lambda v = 0, \quad \Delta_s v \equiv \frac{v_{\phi\phi}}{\sin^2(\theta)} + \frac{1}{\sin\theta}(\sin\theta\, v_\theta)_\theta.$$

The operator Δ_s is the *spherical Laplacian* (which is also known as the Laplace-Beltrami operator for a sphere). Its eigenfunctions will be derived next.

5.1 Eigenfunctions on the surface of a sphere

Letting $v = p(\theta)q(\phi)$, separating variables gives

$$\frac{q''(\phi)}{q(\phi)} + \left(\frac{\sin\theta(\sin\theta\, p'(\theta))'}{p(\theta)} + \lambda\sin^2\theta\right) = 0. \tag{17}$$

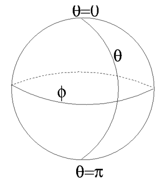

Figure 3: Spherical coordinates used here.

The terms involving ϕ and θ must therefore each equal constants λ_q and λ_p, which sum to 0. The q equation is $q'' + \lambda_q q = 0$, equipped with periodic boundary conditions $q(0) = q(2\pi)$ and $q'(0) = q'(2\pi)$. We therefore immediately know the solutions,

$$q = \cos(m\phi), \sin(m\phi), \quad \lambda_q = m^2, \qquad m = 0, 1, 2, \ldots.$$

If u is complex valued, the eigenfunctions are $q = \exp(im\phi), m = 0, \pm1, \pm2, \ldots$ instead. Then $\lambda_p = -\lambda_q = -m^2$, so the equation for p in (17) reads

$$\frac{1}{\sin\theta}(\sin\theta\, p')' + \Big(\lambda - \frac{m^2}{\sin^2\theta}\Big)p = 0.$$

This appears very unfavorable, but a lucky change of variables $s = \cos\theta$ gives

$$[(1-s^2)p']' + [\lambda - m^2/(1-s^2)]p = 0, \tag{18}$$

which is known as the *associated Legendre's equation*. The boundary conditions on p simply come from the fact that the complete solution must be bounded at $\theta = 0, \pi$. In the s variable, this implies $p(s = \pm1)$ is bounded.

First set $m = 0$. As with the Bessel equation, we look for a solution of the form

$$p(s) = \sum_{k=0}^{\infty} a_k s^k.$$

Substituting into the equation, we have

$$\sum_{k=2}^{\infty}(k+2)(k+1)a_{k+2}s^k - \sum_{k=0}^{\infty}(k^2+k-\lambda)a_k s^k = 0.$$

Equating coefficients of s^k leads to the recursion relation

$$a_{k+2} = a_k\frac{k(k+1)-\lambda}{(k+2)(k+1)}.$$

Notice if $\lambda = l(l+1)$ for some $l = 0, 1, 2, 3, \ldots$ something remarkable happens: the series solution has zero coefficients for $k > l$, provided $a_1 = 0$ if l is even and $a_0 = 0$ if l is odd. These solutions are called *Legendre polynomials*, denoted $P_l(s)$. Since they are bounded at $s = \pm1$, they are admissible as eigenfunctions of (18) with $m = 0$, where the corresponding eigenvalues are $\lambda = l(l+1)$. The first three are $P_0 =$, $P_1 = s$ and $P_2 = \frac{1}{2}(3s^2 - 1)$.

What if $\lambda \neq l(l+1)$ for some positive integer l? The ratio of successive coefficients $|a_{k+2}/a_k|$ in the power series goes to one, and therefore the power series has an interval of convergence $(-1, 1)$. This means that the series must diverge at $s = \pm1$, which can't happen since the eigenfunctions must be bounded there.

Now for $m > 0$. It turns out the solution of the associated Legendre equation (18) can be written (remarkably!) in terms of the Legendre polynomials themselves by

$$p(s) = P_l^m(s) \equiv (1-s^2)^{m/2} \frac{d^m}{ds^m} P_l(s).$$

The functions $P_l^m(s)$ are called the *associated Legendre functions*. Notice that if $m > l$, the right hand side is zero since P_l is a polynomial. Thus we have a funny enumeration scheme for the eigenfunctions, specifically $l = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots, \pm l$. The first several are given in table 1.

l	m	$P_l^{\lvert m \rvert}$	Y_l^m
0	0	1	1
1	0	$s = \cos\theta$	$\cos\theta = z$
1	± 1	$(1-s^2)^{1/2} = \sin\theta$	$e^{\pm i\phi}\sin\theta = \frac{x \pm iy}{r}$
2	0	$\frac{1}{2}(3s^2-1) = \frac{1}{2}(3\cos^2\theta - 1)$	$\frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{2}\frac{2z^2-x^2-y^2}{r^2}$
2	± 1	$3s(1-s^2)^{1/2} = 3\cos\theta\sin\theta$	$3e^{\pm i\phi}\cos\theta\sin\theta = 3\frac{(x\pm iy)z}{r^2}$
2	± 2	$3(1-s^2) = 3\sin^2\theta$	$3e^{\pm 2i\phi}\sin^2\theta = 3\frac{(x\pm iy)^2}{r^2}$

Table 1: Some of the associated Legendre functions $P_l^m(s)$, where $s = \cos\theta$, and corresponding spherical harmonics $Y_l^m(\phi, \theta)$.

Now let's put it all together. The complete set of eigenfunctions is

$$\cos(m\phi)P_l^m(\cos\theta), \quad \sin(m\phi)P_l^m(\cos\theta), \quad l = 0, 1, 2, \ldots \quad m = 0, 1, 2, \ldots, l,$$

with corresponding eigenvalues $\lambda = l(l+1)$. Often, the eigenfunctions are written in complex form, which gives

$$Y_l^m(\phi, \theta) = e^{im\phi}P_l^m(\cos\theta), \quad l = 0, 1, 2, \ldots \quad m = 0, \pm 1, \pm 2, \ldots, \pm l.$$

These are the famous *spherical harmonics*, the first several of which are given in table 1. Notice that for each eigenvalue $l(l+1)$, there are $2n+1$ linearly independent eigenfunctions. The set Y_l^m, $-l \le m \le l$ forms one such set, but others are possible.

As with Bessel functions, a lot of complexity can be hidden inside the notation Y_l^m. The details of these eigenfunctions is not always needed: for example, one computes $\Delta_s Y_l^m = -l(l+1)Y_l^m$ without using any facts about Legendre functions, simply because the spherical harmonics are eigenfunctions of Δ_s. Additionally, the operator Δ_s is (unsurprisingly) self-adjoint if one uses the inner product

$$\langle v_1, v_2 \rangle = \int_0^{\pi} \int_0^{2\pi} v_1(\phi, \theta) v_2(\phi, \theta) \sin\theta \, d\phi \, d\theta. \tag{19}$$

It follows that $< Y_l^m, Y_{l'}^{m'} >= 0$ unless $l = l'$ and $m = m'$ by orthogonality.

Figure 4 shows a visualization of some of the spherical harmonics. Their nodal lines are curves on the surface of a sphere, and intersect orthogonally much like the eigenfunctions for the rectangle.

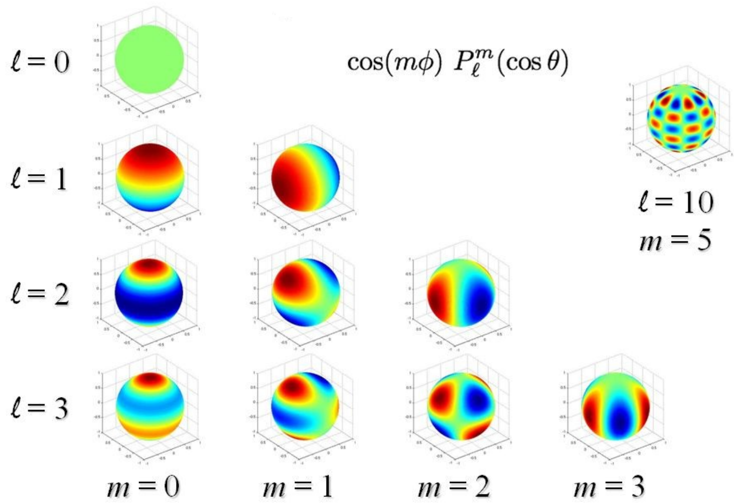

Figure 4: Visualization of some of the spherical harmonics

5.2 Laplace's equation inside the sphere

We return to equation (16), which is to be solved on the interior of a sphere of radius a, subject to a boundary condition

$$u(a, \phi, \theta) = f(\phi, \theta).$$

Recall that separation of variables $u = R(r)v(\phi, \theta)$ produced

$$\frac{-r^2R'' - 2rR'}{R} = \frac{\Delta_s v}{v} = -\lambda$$

thus $\Delta_s v + \lambda v = 0$. We immediately have a solution of the eigenvalue problem:

$$v = Y_l^m(\phi, \theta), \quad \lambda = l(l+1), \quad l = 0, 1, 2, \ldots \quad m = 0, \pm 1, \pm 2, \ldots, \pm l.$$

(even if $f(\phi, \theta)$ is real, it is more convenient to use the complex form for the eigenfunctions). For each eigenvalue $\lambda = l(l+1)$, the radial equation is just

$$r^2R'' + 2rR' - l(l+1)R = 0. \tag{20}$$

This is an Euler equation with solutions of the form $R = r^\alpha$. Substituting into (20) gives

$$\alpha(\alpha - 1) + 2\alpha - l(l+1) = (\alpha - l)(\alpha + l + 1) = 0,$$

so that $\alpha = l$ or $\alpha = -l - 1$. For domains which contain the origin, solutions with $\alpha < 0$ must be rejected.

A general solution is given by a superposition of separated solutions

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_l^m(\phi, \theta),$$

where A_{lm} can be complex valued. We now set $r = a$ and use the boundary condition to give

$$f(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l Y_l^m(\phi, \theta).$$

This is just an orthogonal expansion, and the coefficients are determined in the usual way by taking inner products with each eigenfunction using (19). This leads to

$$A_{lm} = \frac{1}{a^l} \frac{\langle f, Y_l^m \rangle}{\langle Y_l^m, Y_l^m \rangle}, \quad \langle v_1, v_2 \rangle = \int_0^{\pi} \int_0^{2\pi} v_1(\phi, \theta) v_2(\phi, \theta) \sin\theta d\phi d\theta.$$

5.3 Spherical harmonics in retrospect

It seems like a lot of guesswork went into deriving spherical harmonics, and their characterization is still somewhat hidden behind the mystery of the Legendre functions. Remarkably, in terms of Cartesian coordinates, spherical harmonics are easy to understand!

To see why this is the case, let $P(\mathbf{x}) = r^l Y_l^m(\phi, \theta)$ be a separated solution to Laplace's equation, where $\mathbf{x}$ is a vector coordinate for (r, ϕ, θ). $P(\mathbf{x})$ has a special property – it is *homogeneous of degree* l, which means

$$P(\alpha \mathbf{x}) = \alpha^l P(\mathbf{x}), \quad \alpha > 0.$$

A well known theorem about homogeneous functions is a follows:

If $P(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is continuous and homogeneous of degree l, then $P(\mathbf{x})$ is a polynomial of degree l.

(the proof is easy: just differentiate $P(\alpha\mathbf{x}) = \alpha^l P(\mathbf{x})$ with respect to α exactly $l+1$ times). It follows that

$$Y_l^m = \frac{P(\mathbf{x})}{r^l}, \quad P(\mathbf{x}) \text{ is a homogeneous polynomial of degree } l \text{ which solves Laplace's equation.}$$

For $l = 0$, the only possibility is $P = 1$ and $Y_0^0 = 1$. For $l = 1$, there are three linearly independent choices for homogeneous polynomials of degree one: $P = x, y, z$. These lead to z/r, which is Y_1^0, and x/r and y/r which are the real and imaginary parts of Y_1^1.

The converse of the idea above is also true:

> If $P(\mathbf{x})$ is a homogeneous polynomial of degree l which solves Laplace's equation, then $P(\mathbf{x}/r)$, where $r = |\mathbf{x}|$, is an eigenfunction of Δ_s with eigenvalue $l(l+1)$.

Since P is homogeneous of degree l, then $P(\mathbf{x}/r) = P(\mathbf{x})/r^l$ or $P(\mathbf{x}) = r^l P(\mathbf{x}/r)$. Plugging $u = r^l P(\mathbf{x}/r)$ into (16) gives

$$[l(l-1) + 2l] r^{l-2} P(\mathbf{x}/r) + r^{l-2} \Delta_s P(\mathbf{x}/r) = 0, \tag{21}$$

so that $\Delta_s P(\mathbf{x}/r) + l(l+1) P(\mathbf{x}/r) = 0$ as claimed. Note that in general, $P(\mathbf{x}/r)$ will not be one of the "standard" eigenfunctions Y_l^m, but rather a linear combination of them. For example, it is easy to see that $P = xz$ is a homogeneous polynomial of degree 2 and $\Delta P = 0$. It follows that $P(\mathbf{x}/r) = xy/z^2$ is an eigenfunction with eigenvalue $2(2+1) = 6$. It happens to be a linear combination of the spherical harmonics

$$\frac{xz}{r^2} = \frac{1}{6}(Y_2^{-1} + Y_2^1).$$