Quasi-linear equations: shocks and rarefactions

Consider the the first order "quasi-linear" equation and initial condition

$$u_t + c(u)u_x = 0, \quad u(x,0) = f(x) \quad -\infty < x < \infty \tag{1}$$

If we knew u to begin with, then we could find the characteristics by solving $X'(T) = c(u(X(T),T))$, $X(t) = x$. Since along characteristics $u = U(T)$ is a constant, the solution must be $u(x,t) = u(X(0),0) = f(X(0))$. But since we don't know the solution u in advance, how are we to find the initial position of the characteristic $X(0)$? The answer is, we must find the characteristic $X(T)$ and the solution value on the characteristic U *simultaneously*, in a way that is compatible with the initial condition.

Example. Suppose we want to solve $u_t + 3uu_x = 0$ with $u(x,t) = x$. Characteristics solve $X'(T) = 3U$ which is a constant, subject to $X(t) = x$. The resulting solution is

$$X(T) = 3U(T-t) + x.$$

Then the starting point of the characteristic is $X(0) = x - 3Ut$, and at that point the solution value must match the initial condition

$$U = u(X(0),0) = X(0) = x - 3Ut.$$

This equation gives U implicitly, and solving for U gives the desired solution

$$u(x,t) = U = \frac{x}{1+3t}.$$

Since the solution is constant on each characteristic for equations of the form (1), each characteristic has constant speed, which means that each one is a straight line. This creates two problems. The first is that characteristics will intersect if their speed to the left (on the x axis) is greater than their speed to the right. This leads to ambiguity as to how to assign the solutions value. The second problem is that characteristics emanating from the $t = 0$ axis don't have to pass through every point (x,t) for $t > 0$. Let's look at each case separately.

1 Shocks

Consider the problem

$$u_t + uu_x = 0, u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases} \tag{2}$$

The characteristics which begin where $x < 0$ have speed 2, whereas those that begin where $x > 0$ have speed 1. Thus if $t < x < 2t$, there are two different characteristics which terminate at (x,t).

The resolution of this issue is to allow the solution to be *discontinuous*,

$$u(x,t) = \begin{cases} 2 & x < x_s(t) \\ 1 & x > x_s(t) \end{cases}$$

where $x_s(t)$ is a curve which is called a *shock*. Discontinuous solutions of partial differential equations is a familiar idea from the discussion of Green's functions, and does not pose a problem if derivatives are interpreted in the sense of distributions. For a moving discontinuity, both u_t and u_x are delta functions and the speed is chosen so that they balance in equation (1).

1.1 Shock speed

The moving discontinuity $x_s(t)$ will in general solve an initial value problem of the form $x_s' = c_s(x_s)$ where $x_s(0) =$ initial location of shock. To find the shock speed $c_s(x_s)$, it is useful to first rewrite (1) in *conservation form*

$$u_t + J(u)_x = 0, \quad J'(u) = c(u). \tag{3}$$

Assume first there is a smooth "step-like" solution traveling at speed c_s of the form $u = h(x - c_s t)$ with

$$h \to \begin{cases} u_L & x \to -\infty \\ u_R & x \to +\infty. \end{cases}$$

Plugging this into the equation, we find

$$-c_s h' + J(h)' = 0, \tag{4}$$

which integrated from $-\infty$ to ∞ gives $-c_s(u_R - u_L) + J(u_R) - J(u_L) = 0$ or

$$c_s = \frac{J(u_R) - J(u_L)}{u_R - u_L}. \tag{5}$$

This gives us a formula for the speed of the shock known as the *Rankine-Hugoniot* condition. In general, u_L and u_R are known in advance; they are simply the usual method of characteristics solutions on either side of the shock, evaluated at x_s. For the above example, $J(u) = u^2/2$, and therefore the shock speed is $dx_s/dt = [J(2) - J(1)]/(2-1) = 3/2$. To be consistent with the initial condition $x_s(0) = 0$, so that $x_s(t) = 3/2t$.

2 Rarefactions

Consider a modification of the above example problem

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases} \tag{6}$$

The characteristics which begin where $x < 0$ have speed 1, whereas those that begin for $x > 0$ have speed 2. Thus if $t < x < 2t$, there are no characteristics in this set which terminate at (x,t). This region is called a *rarefaction*.

It turns out that there are an infinite number of ways to specify u in this region to get a solution. This is problematic, and points to failure of the model itself. In these circumstances, one needs an extra condition to "pick out" the most meaningful solution. The principle we will rely on is often called the entropy condition, which can be stated informally as

"Characteristics cannot emerge from other characteristics."

This means that the characteristics in the rarefaction (which are still lines) must all start at the point of discontinuity $x = 0$, and they fan out as time increases.

The issue still remains what value of u to assign to each characteristic. Since the speed of characteristics is related to u, the answer is obvious: in a rarefaction, $u(x,t)$ needs to be made consistent with the speed of the characteristic which goes through (x,t). Rarefaction characteristics will solve $X'(T) = c(U) =$ a constant, with two side conditions

$$X(t) = x, \quad X(0) = x_r, \quad \text{where } x_r \text{ is the starting point of rarefaction fan}$$

Imposing both conditions leads to an implicit equation for U.

In the above example, rarefaction characteristics solve $X'(T) = U =$ a constant, subject to $X(0) = 0$ and $X(t) = x$. Imposing the initial condition first, one gets $X(T) = UT$. On the other hand, letting $T = t$ gives $x = Ut$ which implies $u(x,t) = Ux/t$. The complete solution is therefore

$$u(x,t) = \begin{cases} 1 & x < t \\ x/t & t < x < 2t \\ 2 & x > 2t. \end{cases}$$

3 Examples

Example (non-constant shock speed). Consider

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} 0 & x < 0 \\ 2x & 0 < x < 1 \\ 0 & x > 1. \end{cases}$$

There are three regions to consider. The first is where $x < 0$, in which case all characteristics have zero speed and the solution is just $u = 0$. The second region is where $0 < x < x_s(t)$, which has characteristics with monotonically increasing speed in x, and where the shock location x_s is yet to be determined. The third region is where $x > x_s(t)$, where again characteristics have zero speed and $u = 0$.

In the second region where $0 < x < x_s(t)$, we have $X'(T) = U$ subject to $X(t) = x$, which has a solution

$$X(T) = U(T - t) + x, \quad \text{so that } X(0) = x - Ut.$$

Since $U = u(X(0), 0) = 2X(0)$, it follows that $2(x - Ut) = U$ or $u(x,t) = U = 2x/(1 + 2t)$.

Writing the equation in conservation form as $u_t + J(u)_x = 0$ with $J = u^2/2$, the shock location $x_s(t)$ evolves according to

$$x_s'(t) = \frac{J(2x_s/(1+2t)) - J(0)}{2x_s/(1+2t) - 0} = \frac{x_s}{1+2t}.$$

This is a differential equation for $x_s(t)$, with initial condition $x_s(0) = 1$. The solution after separating variables $dx_s/x_s = dt/(1 + 2t)$ is

$$x_s(t) = \sqrt{1 + 2t}.$$

Example (shock creation) Consider

$$u_t + (u+1)u_x = 0, \quad u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & x > 1. \end{cases}$$

Notice that characteristics which start at $x = 0$ and $x = 1$ have respective speeds 2 and 1 and therefore are described by the lines $X_1(T) = 2T$ and $X_2(T) = 1 + T$. These intersect when $X = 2$ and $T = 1$, at which point a shock must be created, whose velocity is

$$x_s'(t) = \frac{J(0) - J(1)}{0 - 1}, \quad J(u) = u^2/2 + u,$$

so that $x_s' = 3/2$. Since $x_s(1) = 2$, $x_s(t) = 3/2(t-1) + 2$.

On the other hand, before time $t = 1$, there is no shock at all. For the region $X_1(t) < x < X_2(t)$, the characteristics with $X(t) = x$ all have $0 < X(0) < 1$, and satisfy $X'(T) = (U + 1)$. They are therefore the lines $X(T) = (U+1)(T-t) + x$. To be consistent with the initial condition,

$$U = u(X(0), 0) = 1 - [x - (U+1)t],$$

so that

$$u(x,t) = U = \frac{1 - x + t}{1 - t}.$$

Notice that this solution breaks down at $t = 1$, which is to be expected since this is when the shock wave forms.

Example (multiple shocks). Consider

$$u_t + 3u^2 u_x = 0, \qquad u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & 1 < x < 2 \\ 1 & x > 2. \end{cases}$$

At least when t is small, there must be three regions where the characteristic speeds are

$$X'(T) = 3U^2 = \begin{cases} 27 & x < x_1(t) \\ 12 & x_1(t) < x < x_2(t) \\ 3 & x > x_2(t). \end{cases}$$

There are therefore two shocks with locations $x_1(t), x_2(t)$. Since conservation form $u_t + J(u)_x = 0$ for this equation has $J(u) = u^3$, they satisfy initial value problems

$$x_1'(t) = \frac{2^3 - 3^3}{2 - 3} = 19, \qquad x_1(0) = 1$$

and

$$x_2'(t) = \frac{1^3 - 2^3}{1 - 2} = 7, \qquad x_2(0) = 2,$$

so that

$$x_1(t) = 19t + 1, \quad x_2(t) = 7t + 1.$$

These two lines intersect where $t = 1/12$ and $x = 31/12$, and after this point there is only one shock, denoted $x_3(t)$, with $u = 3$ on the left and $u = 1$ on the right. This shock evolves according to

$$x_3'(t) = \frac{1^3 - 3^3}{1 - 3} = 13, \qquad x_3(1/12) = 32/12,$$

so that $x_3(t) = 13(t - 1/12) + 31/12$.