Separation of Variables

A typical starting point to study differential equations is to guess solutions of a certain form. Since we will deal with linear PDEs, the superposition principle will allow us to form new solutions from linear combinations of our guesses, in many cases solving the entire problem. To begin with, we will consider functions of two variables $u(v_1, v_2)$ (for example $u(x, y)$ or $u(r, \theta)$), where the domain is very particular: it must be of the form $(v_1, v_2) \in [a, b] \times [c, d]$. It will also be necessary to have homogeneous boundary conditions on opposite boundaries $v_1 = a$ and $v_1 = b$ (or alternatively $v_2 = c$ and $v_2 = d$).

0.1 The separation principle

Suppose we have a problem with variables $x_1, x_2, \ldots, x_n$, and let $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ be functions of one variable each. If it happens that

$$f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n) = 0, \quad \text{for all } (x_1, x_2, \ldots, x_n) \in \Omega \tag{1}$$

then each term in (1) is a constant. This is easy to show: just take partial derivatives of the left hand expression with respect to each x_i. This gives $f_i'(x_i) = 0$ so $f_i(x_i) = \lambda_i$; each λ_i are called *separation constants*. The point of separation of variables is to get to equation (1) to begin with, which can be done for a good number of homogeneous linear equations.

1 The wave equation

As a first example, consider the wave equation with boundary and initial conditions

$$u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0 = u(L, t), \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \tag{2}$$

We attempt an educated guess: find solutions of the form $u(x, t) = X(x)T(t)$ which satisfy everything except the inhomogeneous initial conditions. These will be called *separated solutions*. Of course, not every solution will be found this way, but we have a trick up our sleeve: the superposition principle guarantees that linear combinations of separated solutions will also satisfy both the equation and the homogeneous boundary conditions. The proper choice of linear combination will allow for the initial conditions to be satisfied.

Inserting $u(x, t) = X(x)T(t)$ into the equation in (2) gives $XT'' = c^2 TX''$. We can separate the x- and t- dependence by dividing to give

$$\frac{T''}{c^2 T} = \frac{X''}{X}.$$

Using the separation principle, it follows that the left and right hand sides are equal to a constant, which we will call $-\lambda$. Similarly, inserting $u(x, t) = X(x)T(t)$ into the boundary conditions in (2) means that $X(0)$ and $X(L)$ must be zero. We therefore get two ODEs: a boundary value problem for X

$$X'' + \lambda X = 0, \quad X(0) = 0 = X(L), \tag{3}$$

and an unconstrained equation for T,

$$T'' + c^2 \lambda T = 0. \tag{4}$$

Notice immediately that the problem for X is actually an eigenvalue problem which was solved previously. We have a countable number of solutions

$$X_n = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

Equation (4) has two linearly independent solutions: $\sin(c\sqrt{\lambda}t)$ and $\cos(c\sqrt{\lambda}t)$. Setting $\lambda = \lambda_n$ for each n, we find that separated solutions have the form

$$\sin(cn\pi t/L)\sin(n\pi x/L), \quad \cos(cn\pi t/L)\sin(n\pi x/L), \quad n = 1, 2, 3, \ldots.$$

We might hope that a linear combination of separated solutions solves the whole problem,

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(cn\pi t/L) + B_n\sin(cn\pi t/L)]\sin\left(\frac{n\pi x}{L}\right). \tag{5}$$

Using the initial conditions in (2) means that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad \psi(x) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right) B_n \sin\left(\frac{n\pi x}{L}\right).$$

These are simply Fourier sine series, and determining the coefficients is just a matter of taking inner products of both sides with the orthogonal eigenfunctions $X_n = \sin(n\pi x/L)$. This gives

$$A_n = \frac{\langle \phi, X_n\rangle}{\langle X_n, X_n\rangle}, \quad B_n = \left(\frac{L}{n\pi c}\right)\frac{\langle \psi, X_n\rangle}{\langle X_n, X_n\rangle}, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product.

At this point, the result (5) may seem anti-climatic. It's difficult to imagine exactly what a superposition of an infinite number of oscillations might even look like. It turns out that we are often more interested in the individual components of the solution, the separated solutions whose spatial and temporal structure is easy to understand. The eigenfunctions that make up the spatial dependence are often called *modes* (or normal modes) whose shape defines the underlying waves. Each mode has its own frequency of oscillation,

$$\omega_n = \frac{n\pi c}{L}, \quad n = 1, 2, 3, \ldots$$

In many problems, in fact, the set of frequencies $\{\omega_n\}$ is much more interesting than the complete solution (5)! These frequencies form the basis for the description of many physical phenomenon, including the production of sound waves and atomic spectra.

2 The diffusion and Laplace equations

The preceding strategy can be immediately adapted to other linear equations with the same domain and boundary conditions such as the diffusion equation

$$u_t = Du_{xx}, \quad u(0,t) = 0 = u(L,t), \quad u(x,0) = \phi(x), \tag{7}$$

and Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = 0 = u(L,y), \quad u(x,0) = h(x), \quad u(x,H) = g(x), \tag{8}$$

For the diffusion equation (7), again set $u = X(x)T(t)$ and separate the variables to give

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

Remarkably, we obtain exactly the same eigenvalue problem for X and λ as before (3). The difference is in the equation for T, which reads

$$T' = -D\lambda T,$$

which has one linearly independent solution $T = \exp(-D\lambda t)$. The separated solutions are therefore

$$\exp(-D(n\pi/L)^2 t)\sin(n\pi x/L), \quad n = 1, 2, 3, \ldots$$

In this case, the modes X_n decay in time rather than oscillate. Note that we had predicted exactly this when discussing conserved and dissipated quantities for the wave and diffusion equations.

A linear combination of separated solutions is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp(-D(n\pi/L)^2 t)\sin\left(\frac{n\pi x}{L}\right). \tag{9}$$

Invoking the initial condition, it follows that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

which means the coefficients are the same as in (6).

For the Laplace equation (8), separation of variables $u = X(x)Y(y)$ leads to $-Y''/Y = X''/X = -\lambda$. We again get the the same eigenvalue problem (3) for X and an equation for Y of the form

$$Y'' = \lambda Y.$$

Since λ is always positive, there are two linearly independent solutions $Y = \exp(\sqrt{\lambda}y)$ and $Y = \exp(-\sqrt{\lambda}y)$. The separated solutions are therefore

$$\exp(n\pi y/L)\sin(n\pi x/L), \quad \exp(-n\pi y/L)\sin(n\pi x/L), \quad n = 1, 2, 3, \ldots$$

A superposition of these is

$$u(x,y) = \sum_{n=1}^{\infty}[A_n \exp(n\pi y/L) + B_n \exp(-n\pi y/L)]\sin\left(\frac{n\pi x}{L}\right). \tag{10}$$

We now try to satisfy the inhomogeneous boundary conditions in (8). Setting $y = 0$ and $y = H$ gives

$$h(x) = \sum_{n=1}^{\infty}(A_n + B_n)\sin\left(\frac{n\pi x}{L}\right), \; g(x) = \sum_{n=1}^{\infty}[A_n \exp(n\pi H/L) + B_n \exp(-n\pi H/L)]\sin\left(\frac{n\pi x}{L}\right).$$

Each represents a Fourier sine series, so upon taking inner products with the eigenfunctions $X_n = \sin(n\pi x/L)$, one gets

$$A_n + B_n = \frac{\langle h, X_n\rangle}{\langle X_n, X_n\rangle}, \quad A_n \exp(n\pi H/L) + B_n \exp(-n\pi H/L) = \frac{\langle g, X_n\rangle}{\langle X_n, X_n\rangle}$$

For each value of n, this represents a system of two equations for the two unknowns A_n, B_n, which can be solved in principle.

Note that we could have used different linearly independent solutions for $Y'' = \lambda Y$, such as $Y = \cosh(\sqrt{\lambda}y)$ and $Y = \sinh(\sqrt{\lambda}y)$, so that the solution reads

$$u(x,y) = \sum_{n=1}^{\infty}[C_n \cosh(n\pi y/L) + D_n \sinh(n\pi y/L)] \sin\left(\frac{n\pi x}{L}\right). \tag{11}$$

The advantage this confers is that the system for coefficients C_n, D_n is readily decoupled:

$$C_n = \frac{\langle h, X_n\rangle}{\langle X_n, X_n\rangle}, \quad C_n \cosh(n\pi H/L) + D_n \sinh(n\pi H/L) = \frac{\langle g, X_n\rangle}{\langle X_n, X_n\rangle}$$

3 Laplace's equation in polar coordinates and Poisson's formula

If the domain happens to have circular geometry (a disk, wedge or annulus) it makes sense to use polar coordinates because the boundaries are just where r or θ are constant. In polar coordinates, the Laplace operator is

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}.$$

For a disk-shaped domain, the problem we want to solve is

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0, \quad u(a,\theta) = h(\theta). \tag{12}$$

This seems different than the previous examples because there are fewer boundary conditions. There are actually hidden boundary conditions when using polar coordinates. The first is that the solution should be finite at $r = 0$; we will note that some of our separated solutions do not have this property. The second is that solutions should be 2π-periodic in θ, since $\theta = 0$ and $\theta = 2\pi$ are the same coordinate.

Separating variables $u = R(r)\Theta(\theta)$ gives $R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = 0$ or after multiplying by $r^2/(R\Theta)$,

$$\frac{\Theta''}{\Theta} = \frac{-r^2R'' - rR'}{R} = -\lambda. \tag{13}$$

Since $u(r,\theta)$ is 2π periodic, Θ and its derivatives should be also. This is now familiar ground: the eigenvalue problem to be solved for Θ is

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi).$$

The boundary conditions are considered homogeneous since any linear combination of periodic solutions will also be periodic. If $\lambda > 0$, it is easiest to write the general solution as $\Theta = \exp(\pm i\sqrt{\lambda}\theta)$ (recall that the real and imaginary parts are also solutions of the ODE). Clearly this is 2π periodic only when $\sqrt{\lambda}$ is a positive integer n. If $\lambda = 0$, the only periodic solution is a constant. Finally, if $\lambda < 0$, solutions are exponentials which can never be periodic. The complete set of eigenvalues and eigenfunctions is therefore

$$\Theta = \begin{cases} 1 & \lambda = 0 \\ \cos(n\theta) & \lambda = n^2 \\ \sin(n\theta) & \lambda = n^2, \end{cases}$$

where $n = 1, 2, 3, \ldots$. These are sometimes called the *circular harmonics* and a linear combination of these forms the standard real Fourier series. (By the way, if we wanted complex-valued solutions for Θ instead, then one would have eigenfunctions $\Theta = \exp(\pm in\theta)$ which comprise the complex Fourier series).

The equation for the radial component in (13) reads

$$r^2R'' + rR' - \lambda R = 0.$$

This is called the *Euler* or *equidimensional* equation, and it is easy to solve! For $\lambda > 0$, solutions are just powers $R = r^\alpha$. Plugging in one gets

$$[\alpha(\alpha-1) + \alpha - \lambda]r^\alpha = 0,$$

so that $\alpha = \pm\sqrt{\lambda}$. If $\lambda = 0$, one can solve for R' first (using separation of variables for ODEs) and then integrating again. This leads to linearly independent solutions 1 and $\ln r$.

We are ready to put together everything we know about separated solutions. Notice that some solutions will not be continuous at the origin – those involving $\ln r$ and $r^{-\alpha}$ – so we must reject them. The remaining separated solutions have the form 1, $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$. Therefore the most general solution we might hope to find is the superposition

$$u = A_0/2 + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)]. \tag{14}$$

Finally, satisfying the boundary condition in (12) means that

$$h(\theta) = A_0/2 + \sum_{n=1}^{\infty} a^n[A_n\cos(n\theta) + B_n\sin(n\theta)].$$

We can see that this is a Fourier series with cosine coefficients a^nA_n and sine coefficients a^nB_n, so that (using the known formulas)

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)d\phi, \quad B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)d\phi.$$

Note this works for A_0 also since we cleverly wrote the first term as $A_0/2$.

Usually, series solutions like (14) cannot be summed analytically, but remarkably it can be done here. Inserting the coefficient formulas into (14) gives

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)[\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)]d\phi.$$

We can use the identity $\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta) = \cos(n(\theta-\phi))$ and reverse the order of summation and integration,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\Big\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\theta-\phi))\Big\}d\phi.$$

The sum is not as bad as it seems; in fact, it is a geometric series in disguise! In particular,

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\theta-\phi)) &= 1 + 2\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\phi)}}{a}\right)^n \\ &= 1 + 2\mathrm{Re}\frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} \\ &= \frac{a^2-r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}. \end{aligned}$$

We therefore have an integral, known as Poisson's formula, for the solution:

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta-\phi)h(\phi)d\phi, \quad P(r,\theta) = \frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\theta)+r^2}. \tag{15}$$

Formula (15) provides a nice interpretation to Laplace's equation. Setting $r = 0$, one immediately obtains

$$u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\theta)d\theta. \tag{16}$$

This says that a solution of Laplace's equation at a point is equal to the average of its values on a circle about that point; this is the *mean value property*. The circle does not have to be the domain boundary, by the way; it simply must reside inside the closure of the problem domain.

A simple consequence of the mean value property is the *maximum principle*. This says that the maximum (and minimum) of solutions to Laplace's equation are obtained on the domain boundary. For example, if $u(x, y)$ represents the steady-state temperature of an object, then the hottest and coldest points can't be in the interior of the object unless the temperature is constant throughout. The maximum principle has many extensions to time-dependent and nonlinear equations.

4 Dealing with inhomogeneous equations or side conditions

Notice that in separating variables for $u(v_1, v_2)$, either the boundaries corresponding to constant v_1 or constant v_2 must have homogeneous conditions to obtain an eigenvalue problem. And if the equation is not homogeneous, it will be impossible to separate variables at all. Luckily, linearity can come to the rescue in these cases. The idea is first find a particular solution which satisfies the equation and/or the inhomogeneous boundary conditions. Then by subtracting off the particular solution, a problem suitable for separation of variables is obtained. The main difficulty is in finding a particular solution; in simple problems, guessing is often effective.

4.1 Inhomogeneous equations

Recall the inhomogeneous superposition principle which says that if u_p is a particular solution satisfying an inhomogeneous equation also satisfied by u, then $w = u - u_p$ satisfies a homogeneous equation. Then it might be the case that the new problem for w is suitable for separation of variables.

Consider the Poisson equation (the inhomogeneous version of the Laplace equation)

$$\Delta u = u_{xx} + u_{yy} = 1, \quad u(0,y) = 0 = u(1,y), \quad u(x,0) = 0 = u(x,1).$$

We must find a particular solution so that $\Delta u_p = 1$, so we guess $u_p = Ax^2 + Bx + C$. Plugging into the equation gives $2A = 1$ or $A = \frac{1}{2}$. We also want u_p to be zero on the lateral boundaries just like Laplace equation problem above. We do this by inserting u_p into the boundary conditions, giving $C = 0$, $B = -\frac{1}{2}$, so that

$$u_p = \frac{1}{2}x(x-1).$$

Now we formulate a problem for $w = u - u_p$ by subtracting each equation and side conditions, which leads to

$$w_{xx} + w_{yy} = 0, \quad w(0,y) = 0 = w(1,y), \quad w(x,0) = -\frac{1}{2}x(x-1) = w(x,1).$$

This problem can now be solved by separation of variables; it is a special case of (8) so we can quote the solution

$$w(x,y) = \sum_{n=1}^{\infty} [A_n \exp(n\pi y) + B_n \exp(-n\pi y)] \sin(n\pi x),$$

where

$$A_n + B_n = -\frac{\langle \frac{1}{2}x(x-1), X_n \rangle}{\langle X_n, X_n \rangle}, \quad A_n \exp(n\pi) + B_n \exp(-n\pi) = -\frac{\langle \frac{1}{2}x(x-1), X_n \rangle}{\langle X_n, X_n \rangle}$$

and from direct computation of integrals,

$$\frac{\langle \frac{1}{2}x(x-1), X_n \rangle}{\langle X_n, X_n \rangle} = -\frac{2}{\pi^4 n^3}[\cos(\pi n) - 1].$$

4.2 Inhomogeneous boundary conditions

In this case we seek a particular solution u_p which satisfies both the equation and inhomogeneous boundary conditions. Then $u - u_p$ will have (at least some) homogeneous boundary conditions and might be suitable for separation of variables.

Suppose we wanted to solve the diffusion equation

$$u_t = Du_{xx}, \quad u(0,t) = u_l, \quad u(L,t) = u_r, \quad u(x,0) = \phi(x). \tag{17}$$

We need a particular solution to the equation u_p which satisfies $u_p(0,t) = u_r$ and $u_p(L,t) = u_l$. The reason for this is that $w = u - u_p$ will also solve the equation but with homogeneous boundary conditions

$$w_t = Dw_{xx}, \quad w(0,t) = 0, \quad w(L,t) = 0, \quad w(x,0) = \phi(x) - u_p. \tag{18}$$

which we have already solved. Notice that for the initial condition to make sense, the particular solution should only be a function of x. A reasonable guess is a linear function $u_p = Ax + B$. Clearly this solves the equation, and if $B = u_l$ and $A = (u_r - u_l)/L$ then the boundary conditions are also satisfied.

As another example, consider the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = p(y), \quad u(L,y) = q(y), \quad u(x,0) = h(x), \quad u(x,H) = g(x), \tag{19}$$

where there are no homogeneous boundary conditions. One needs either $p, q = 0$ or $h, g = 0$ in order to use separation of variables. Therefore, consider the solutions v and w to two separate problems which we have solved before:

$$\begin{aligned} v_{xx} + v_{yy} &= 0, \quad v(0,y) = 0, \quad v(L,y) = 0, \quad v(x,0) = h(x), \quad v(x,H) = g(x), \\ w_{xx} + w_{yy} &= 0, \quad w(0,y) = p(y), \quad w(L,y) = q(y), \quad w(x,0) = 0, \quad w(x,H) = 0. \end{aligned}$$

(Note that when separating variables $w = X(x)Y(y)$, it will be $Y(y)$ that satisfies the eigenvalue problem). Now convince yourself that $v + w$ solves original problem (19).

5 Limitations and extensions to separation of variables

We should acknowledge the limitations of the procedure that is outlined. The first is the essential nature of linearity, which allowed the use the superposition principle. The second is having homogeneous boundary conditions on two opposite sides of the domain. Without these, the boundary value problem would not be an eigenvalue problem at all, and it is unlikely that enough separated solutions could be found to solve the entire problem. The final limitation is the geometry of the domain. In Cartesian coordinates, we are limited to rectangles, and in polar coordinates we are limited to circles, wedges, and annuli.

We shall also consider separation of variables in higher dimensions later on. This will require solution of higher dimensional eigenvalue problems, which will be solved by (you guessed it!) further separation of variables. We will also have the same limitations on geometry: domains in three variables will be limited to rectangular boxes, cylinders and spheres.