Symmetry and similarity solutions

1 Symmetries of partial differential equations

1.1 New solutions from old

Consider a partial differential equation for $u(x,t)$ whose domain happens to be $(x,t) \in \mathbb{R}^2$. It often happens that a transformation of variables gives a new solution to the equation. For example, if $u(x,t)$ is a solution to the diffusion equation $u_t = u_{xx}$, it is easy to show that both $u(x-x_0, t-t_0)$ (where x_0 and t_0 are constants) and $u(-x,t)$ are also solutions. Clearly identifying symmetries is useful, as it allows us to construct new solutions from ones we already know.

1.2 Symmetry transformations

Given a function $u(x,t)$, the independent and dependent variables can be transformed by a mapping $(x,t,u) \to (x',t',u')$, or more specifically $x' = X(x,t,u)$, $t' = T(x,t,u)$ and $u' = U(x,t,u)$ where X, T, U are smooth functions. A partial differential equation is said to have a *symmetry transformation* or *symmetry* if $u'(x,t) = U(x,t,u(X(x,t,u),T(x,t,u)))$ is a solution given that $u(x,t)$ is. In other words, a symmetry transformation maps solutions of an equation to solutions of the same equation.

Let us illustrate this by an example. Consider the equation $u_t = uu_{xx}$, and the transformation $x' = -x$, $t' = -t$ and $u' = -u$. Given a solution $u(x,t)$, it is easy to check that $u'(x,t) = -u(-x,-t)$ is also, simply by plugging it into the equation:

$$\partial_t\Big(-u(-x,-t)\Big) = u_t(-x,-t), \quad \text{and}$$
$$\Big(-u(-x,-t)\Big)\partial_{xx}\Big(-u(-x,-t)\Big) = u(-x,-t)u_{xx}(-x,-t).$$

Often a symmetry transformation takes one or more real parameters. For example, for any $\alpha \neq 0$, it is easy to show that $\alpha^{-2}u(\alpha x)$ is a solution of the Poisson equation $\Delta u = 1$ provided $u(x)$ is. Another common symmetry is translation invariance. This means that if $u(x)$ is a solution in the domain $\mathbb{R}^n$, so is $u(x-x_0)$ for any $x_0 \in \mathbb{R}^n$. Note these are not just a single symmetry transformation, but an entire family.

1.3 Dilation symmetries

The variety of symmetry transformations is potentially vast, so we focus on perhaps the most useful subclass. These involve a transformation of dependent and independent variables by scaling, and they are referred to as *dilation symmetries*.

For functions of two variables, we might expect a rescaling of both variables leads to another solution. For a problem involving space x and time t, we could try

$$x' = \frac{x}{L}, \quad t' = \frac{t}{L^\beta}, \tag{1}$$

where L is any positive parameter. Substitution into the equation of interest usually determines the exponent β, by insisting that the transformed equation is just the same as the original, but evaluated at $(x/L, t/L^\beta)$.

Take the simple transport equation $u_t + cu_x = 0$ as an example. Substituting $u(x/L, t/L^\beta)$ in place of a known solution $u(x,t)$ gives

$$\frac{u_t(x/L, t/L^\beta)}{L^\beta} + c\frac{u_x(x/L, t/L^\beta)}{L} = 0,$$

which is true provided $\beta = 1$. Thus given a solution, say $u = \sin(x - ct)$, we can construct a new solution $u = \sin((x - ct)/L)$.

Symmetry requires that the entire equation can be written as the original after the transformed solution has been inserted. Take for example

$$u_x + y^2 u_y = 0, \tag{2}$$

for which we might try the symmetry $(x, y) \to (x/L, y/L^\beta)$. Substituting $u(x/L, y/L^\beta)$ for u in this equation produces

$$L^{-1}u_x(x/L, y/L^\beta) + L^{-\beta}y^2 u_y(x/L, y/L^\beta) = 0. \tag{3}$$

This is not the same as (2) if $\beta = 1$, since the derivatives are evaluated where y is replaced with y/L^β, but the y^2 coefficient is not. We can fix this problem by rewriting (3) as

$$L^{-1}u_x(x/L, y/L^\beta) + L^{\beta}(y/L^\beta)^2 u_y(x/L, y/L^\beta) = 0. \tag{4}$$

which is true using (2) provided $\beta = -1$.

Some symmetries also involve a rescaling of the dependent variable as well. These can take the form

$$(x, t, u) \to (x/L, t/L^\beta, u/L^\gamma). \tag{5}$$

The exponents β, γ can be found as before, simply by substituting $L^{-\gamma}u(x/L, t/L^\beta)$ into the equation and making sure it is still a solution of the original equation.

Consider as an example the nonlinear convection diffusion equation

$$u_t = uu_{xx} - u_x.$$

Plugging in (5), the three terms in the equation are rescaled as $L^{-\gamma-\beta}$, $L^{-2\gamma-2}$ and $L^{-\gamma-1}$, respectively. It follows that $\gamma + \beta = 2\gamma + 2 = \gamma + 1$, so that $\beta = 1$ and $\gamma = -1$.

Notice that for linear homogeneous equations, the solution can be rescaled by any constant, regardless of how the other variables are scaled. This means that γ in this case is arbitrary, and another condition is needed to fix its value. This inability to fix scaling exponents is called *incomplete similarity*, and is discussed later.

2 Similarity solutions

If a PDE has a symmetry transformation $(x, t) \to (x/L, t/L^\beta)$, then a solution of the PDE of the form $u = f(\eta)$, where the *similarity variable* is $\eta = x/t^{1/\beta}$, is called a *similarity solution*. The practical benefit to the idea of similarity solutions is that the function to be found f has only one independent variable η, and typically satisfies an ordinary differential equation (often with additional boundary conditions). There is, however, no guarantee that such a solution exists.

We note that similarity solutions are invariant under the symmetry transformation:

$$u(x/L, t/L^\beta) = f\left(\frac{x/L}{(t/L^\beta)^{1/\beta}}\right) = f\left(\frac{x}{t^{1/\beta}}\right) = u(x, t).$$

It turns out that the converse is true: *every* solution which is invariant under the symmetry transformation must be a function of the similarity variable η alone. This can be demonstrated with the one-to-one change of variables

$$\eta = \frac{x}{t^{1/\beta}}, \quad \xi = xt.$$

(technically this only works for positive x and t, but a similar argument can be made when they are negative). Suppose now that there is a solution which is invariant under the symmetry transformation, which is written in the new coordinates as $w(\eta, \xi)$. The transformation in terms of the new variables is $(\eta, \xi) \to (\eta, \xi/L^{\beta+1})$, so that

$$w(\eta, \xi) = w(\eta, \xi/L^{\beta+1}).$$

Differentiating this expression with respect to L and setting $L = 1$ gives

$$w_\xi(\eta, \xi/L^{\beta+1})\xi = 0,$$

which means that w must be a function of η alone, as claimed.

Here is an example of a similarity solution for the diffusion equation

$$u_t = Du_{xx}, \quad -\infty < x < \infty. \tag{6}$$

We check that $u(x/L, t/L^\beta)$ is another solution by substituting into (6), from which we find that $\beta = 2$. We therefore seek a similarity solution of the form $u = f(\eta)$ where $\eta = x/\sqrt{t}$. Substitution into (6) gives

$$Df''(\eta) + \frac{\eta}{2}f'(\eta) = 0.$$

This can be integrated once (by separating variables) to give

$$f'(\eta) = Ce^{-\eta^2/4D},$$

and therefore

$$f(\eta) = C_1\text{erf}(\eta/4D) + C_2, \quad \text{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy, \tag{7}$$

where C, C_1, C_2 are at this point arbitrary constants. The solution (7) does not decay at infinity. To find such a solution, a more general set of symmetries needs to be considered.

2.1 Similarity solutions with general dilation transformations

Similarity solutions can also be found for the more general dilation symmetry (5). As before, these are solutions which are invariant under transformation. In this case, they have the form

$$u = t^{-\gamma/\beta}f(\eta), \quad \eta = x/t^{1/\beta}$$

since

$$L^{-\gamma}u(x/L, t/L^\beta) = L^{-\gamma}(t/L^\beta)^{-\gamma/\beta}f\left(\frac{x/L}{(t/L^\beta)^{1/\beta}}\right) = t^{-\gamma/\beta}f\left(\frac{x}{t^{1/\beta}}\right) = u(x, t).$$

For example, consider the nonlinear diffusion equation

$$u_t = uu_{xx} - u^3. \tag{8}$$

Inserting $L^{-\gamma}u(x/L, t/L^\beta)$ in place of u, we have

$$\frac{1}{L^{\gamma+\beta}}u(x/L, t/L^\beta) = \frac{1}{L^{2\gamma+2}}u(x/L, t/L^\beta)u_{xx}(x/L, t/L^\beta) - \frac{1}{L^{3\gamma}}u(x/L, t/L^\beta)^3. \tag{9}$$

To ensure this is true, we need $\gamma + \beta = 2\gamma + 2 = 3\gamma$, which implies $\gamma = 2$ and $\beta = 4$. Therefore a similarity solution will have the form

$$u = t^{-1/2}f(\eta), \quad \eta = x/t^{1/4}.$$

Plugging into (8) gives an ODE

$$-\frac{1}{2}f - \frac{\eta}{4}f' = ff'' - f^3.$$

In cases like these, numerical methods are needed since there are no analytical techniques for solving the ODE.

2.2 Incomplete similarity

It often happens that scaling exponents for (5) are not uniquely determined; this situation is called *incomplete similarity*. Extra conditions, such as boundary conditions or conservation laws, are required to fix the value of the exponents. In some cases, the exponents must be determined simultaneously with the similarity solution; this type of problem is often called a nonlinear eigenvalue problem.

Let's investigate symmetries of the form (5) for the diffusion equation (6). After inserting $L^{-\gamma}u(x/L, t/L^\beta)$ into the equation, the exponents of L will match provided $\gamma + \beta = \gamma + 2$. Thus $\beta = 2$ but γ is undetermined.

Suppose we want a similarity solution of the form $u = t^\alpha f(x/t^{1/\beta})$, where $u \to 0$ as $x \to \pm\infty$ (α here plays the role of $-\gamma/\beta$). We have already found a similarity solution for the case $\alpha = 0$, but it does not decay to zero at infinity. Notice that if u decays at infinity, then there is a conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)dx = 0.$$

Inserting the similarity solution and changing variables in the integral implies

$$\frac{d}{dt}\int_{-\infty}^{\infty} t^{\alpha+1/\beta}f(\eta)d\eta = 0,$$

so it must be that $\alpha = -1/\beta = -1/2$. Now substituting $u = t^{-1/2}f(x/t^{1/2})$ into (6), we get an ODE

$$f'' + \frac{\eta}{2}f' + \frac{1}{2}f = f'' + \frac{1}{2}(\eta f)' = 0.$$

Using the condition $f(\pm\infty) = 0$, one integration produces the separable equation $f' = -\eta f/2$. Solving this gives $f(\eta) = Ce^{-\eta^2/4}$ for some constant C. Notice this recovers the fundamental solution $u = Ct^{-1/2}e^{-x^2/4t}$ if C is chosen so that $\int_{-\infty}^{\infty} u\,dx = 1$.

2.3 Examples of similarity solutions

Example 1. Consider the nonlinear transport equation

$$u_t + uu_x = 0.$$

Looking for a symmetry of the form (1) gives $\beta = 1$, so one might try a similarity solution of the form $u = f(x/t)$. Plugging in gives

$$-xt^{-2}f' + t^{-1}ff' = 0$$

or

$$f'(f - \eta) = 0.$$

There are two possibilities: either $f' = 0$, or $f = \eta$. The latter leads to the solution

$$u(x,t) = \eta = \frac{x}{t}.$$

This is in fact a rarefaction wave that fills all of space.

Example 2. A model for a convective thermal layer is

$$3yu_x - u_{yy} = 0, \quad u(0,y) = 1, \quad u(x,0) = 0. \tag{10}$$

In this case, we can treat x like the time variable, and substitute $u(x/L^\beta, y/L)$ into the equation in place of u. This gives

$$\frac{3y}{L^\beta}u_x(x/L^\beta, y/L) - \frac{1}{L^2}u_{yy}(x/L^\beta, y/L) = 0.$$

To make this like equation (10), y should be replaced by y/L everywhere, including the factor in front of u_x. This can be done by writing

$$\frac{3(y/L)}{L^{\beta-1}}u_x(x/L^\beta, y/L) - \frac{1}{L^2}u_{yy}(x/L^\beta, y/L) = 0.$$

It follows that $u(x/L^\beta, y/L)$ is still a solution provided $\beta = 3$.

We can now look for a similarity solution of the form $u = f(\eta)$ where $\eta = y/x^{1/3}$. Inserting into the problem (10) gives

$$-\eta^2 f'(\eta) - f''(\eta) = 0, \quad f(0) = 0, \quad \lim_{\eta\to\infty} f(\eta) = 1.$$

The last two conditions follow from the idea that $\eta \to \infty$ as $x \to 0$ and $\eta \to 0$ as $y \to 0$. The equation can now be integrated twice after separating variables $f''/f' = -\eta^2$, giving

$$f = A\int_0^\eta e^{-s^3/3}ds + B,$$

where A, B are constants of integration. Using the boundary conditions for f, it follows that $B = 0$ and

$$A = \left(\int_0^\infty e^{-s^3/3}ds\right)^{-1}.$$

Example 3. An unwinding fluid vortex can be described by the initial value problem

$$v_t = \left(\frac{1}{r}[rv]_r\right)_r, \quad r > 0, \quad v(r,0) = \frac{1}{r}. \tag{11}$$

Inserting $L^{-\gamma}v(r/L, t/L^\beta)$, one gets

$$\frac{1}{L^{\beta+\gamma}}v_t(r/L, t/L^\beta) = \frac{1}{L^{2+\gamma}}\left(\frac{1}{r}[rv(r/L,t/L^\beta)]_r\right)_r \quad = \frac{1}{L^{2+\gamma}}\left(\frac{1}{(r/L)}[(r/L)v(r/L,t/L^\beta)]_r\right)_r,$$

which is the same as (11) evaluated at $(r/L, t/L^\beta)$ provided $\beta = 2$. This is a situation of incomplete similarity, but the initial condition provides an extra constraint. Substituting $L^{-\gamma}v(r/L, 0)$ into the initial condition gives

$$L^{-\gamma}v(r/L, 0) = L^{-1}(r/L)^{-1},$$

therefore $\gamma = 1$.

One then expects a similarity solution of the form $v = t^{-1/2}f(\eta)$ with $\eta = r/t^{1/2}$. It is easier in the long run, however, to rewrite this as

$$t^{-1/2}f(r/t^{1/2}) = \frac{1}{r}[\eta^{-1}f(r/t^{1/2})].$$

Notice the term in brackets can be written as a single function of x^2/t instead. Therefore we may use a similarity ansatz of the form

$$v = \frac{g(\eta)}{r}, \quad \lim_{\eta \to \infty} f(\eta) = 1.$$

where now $\eta = r^2/t$. With all this established, substitution into (11) gives

$$f' + 4f'' = 0,$$

whose general solution is

$$f(r) = A + Be^{-\eta/4}.$$

Using the boundary condition for f gives $A = 1$. If the complete solution is bounded at $r = 0$ for $t > 0$, this requires $f(0) = 0$ so that $B = -1$. The solution in original variables is therefore

$$v(r,t) = \frac{1}{r}\left(1 - \exp\left(-\frac{r^2}{4t}\right)\right).$$

Example 4. The porous medium equation is

$$u_t = (uu_x)_x. \tag{12}$$

If we look for a generalized symmetry of the form (5), we are led to $\beta = 2+\gamma$. This is an incomplete similarity, and it is necessary to look for a condition which will fix either β or γ. Note that the equation is a conservation law with flux $J = -uu_x$, so that

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)dx = 0, \tag{13}$$

provided the solution vanishes at $x = \pm\infty$. If we have a solution of the form

$$u = t^{-\gamma/\beta}f(\eta), \quad \eta = x/t^{1/\beta}, \tag{14}$$

then

$$\int_{-\infty}^{\infty} u(x,t)dx = t^{(1-\gamma)/\beta}\int_{-\infty}^{\infty} f(\eta)d\eta$$

which using (13) means that $\gamma = 1$ and therefore $\beta = 3$. With these exponents in hand, (14) can be inserted into the PDE, giving

$$-\frac{1}{3}(\eta f)' = (ff')'.$$

Integration gives

$$-\frac{\eta}{3}f = ff' + A,$$

where $A = 0$ if the solution vanishes at $x = \pm\infty$. Further integration gives

$$f = B - \frac{\eta^2}{6}.$$

Notice that f is not positive everywhere, nor is it bounded as we required. It is tempting to only retain the positive part and write the solution as

$$u(x,t) = \begin{cases} t^{-1/3}\left(B - \frac{x^2}{6t^{2/3}}\right) & x^2 < 6Bt^{2/3}, \\ 0 & x^2 > 6Bt^{2/3}. \end{cases}$$

This solution is not smooth, but it can be interpreted as a solution to (12) in a more generalized sense. The constant B is determined by the amount of material present, i.e. the integral in (13).