Asymptotic Methods

Algebraic equations

Regular perturbations

Consider a problem $f(x;\epsilon) = 0$. If the solution set for $\epsilon = 0$ is qualitatively the same as for small, nonzero ϵ, the problem is a regular perturbation of $f(x;0) = 0$.

Regular perturbations

Consider a problem $f(x;\epsilon) = 0$. If the solution set for $\epsilon = 0$ is qualitatively the same as for small, nonzero ϵ, the problem is a regular perturbation of $f(x;0) = 0$.

Example: Solve $x^3 - x + \epsilon = 0$ if $\epsilon \ll 1$.

Regular perturbations

Consider a problem $f(x; \epsilon) = 0$. If the solution set for $\epsilon = 0$ is qualitatively the same as for small, nonzero ϵ, the problem is a regular perturbation of $f(x; 0) = 0$.

Example: Solve $x^3 - x + \epsilon = 0$ if $\epsilon \ll 1$.

Insert predefined expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ into equation and sort terms by powers of ϵ:

$$0 = (x_0^3 - x_0) + \epsilon(3x_0^2 x_1 - x_1 + 1) + \epsilon^2(3x_0^2 x_2 - x_2 + 3x_0 x_1^2) + \ldots$$

Regular perturbations

Consider a problem $f(x;\epsilon) = 0$. If the solution set for $\epsilon = 0$ is qualitatively the same as for small, nonzero ϵ, the problem is a regular perturbation of $f(x;0) = 0$.

Example: Solve $x^3 - x + \epsilon = 0$ if $\epsilon \ll 1$.

Insert predefined expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$ into equation and sort terms by powers of ϵ:

$$0 = (x_0^3 - x_0) + \epsilon(3x_0^2 x_1 - x_1 + 1) + \epsilon^2(3x_0^2 x_2 - x_2 + 3x_0 x_1^2) + \dots$$

Solving order by order gives $x_0^3 - x_0 = 0$ so $x_0 = 0, \pm 1$.

Regular perturbations

Consider a problem $f(x;\epsilon) = 0$. If the solution set for $\epsilon = 0$ is qualitatively the same as for small, nonzero ϵ, the problem is a regular perturbation of $f(x;0) = 0$.

Example: Solve $x^3 - x + \epsilon = 0$ if $\epsilon \ll 1$.

Insert predefined expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$ into equation and sort terms by powers of ϵ:

$$0 = (x_0^3 - x_0) + \epsilon(3x_0^2 x_1 - x_1 + 1) + \epsilon^2(3x_0^2 x_2 - x_2 + 3x_0 x_1^2) + \dots$$

Solving order by order gives $x_0^3 - x_0 = 0$ so $x_0 = 0, \pm 1$.
At $\mathcal{O}(\epsilon)$, $3x_0^2 x_1 - x_1 + 1 = 0$ so that $x_1 = 1/(1 - 3x_0^2)$.
At $\mathcal{O}(\epsilon^2)$, $3x_0^2 x_2 - x_2 + 3x_0 x_1^2 = 0$ so that $x_2 = 3x_0 x_1^2/(1 - 3x_0^2)$.
Notice that there are always three solutions for $\epsilon = 0$.

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Example: Solve $\epsilon x^3 - x + 1 = 0$ for small ϵ.

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Example: Solve $\epsilon x^3 - x + 1 = 0$ for small ϵ.

Using $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ as before, one obtains

$$-x_0 + 1 = 0, \quad -x_1 + x_0^3 = 0, \quad -x_2 + 3x_0^2 x_1 = 0, \ldots$$

So that $x_0 = 1$, $x_1 = 1$ and $x_2 = 3$. But this is insufficient since it does not find the other roots!

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Example: Solve $\epsilon x^3 - x + 1 = 0$ for small ϵ.

Using $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ as before, one obtains

$$-x_0 + 1 = 0, \quad -x_1 + x_0^3 = 0, \quad -x_2 + 3x_0^2 x_1 = 0, \ldots$$

So that $x_0 = 1$, $x_1 = 1$ and $x_2 = 3$. But this is insufficient since it does not find the other roots!

Notice that ϵx^3 was assumed smaller than 1 above, and dominant balance was between the last two terms. Must this always be the case?

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Example: Solve $\epsilon x^3 - x + 1 = 0$ for small ϵ.

Using $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ as before, one obtains

$$-x_0 + 1 = 0, \quad -x_1 + x_0^3 = 0, \quad -x_2 + 3x_0^2 x_1 = 0, \ldots$$

So that $x_0 = 1$, $x_1 = 1$ and $x_2 = 3$. But this is insufficient since it does not find the other roots!

Notice that ϵx^3 was assumed smaller than 1 above, and dominant balance was between the last two terms. Must this always be the case?

Suppose instead dominant balance between first two terms, so that $\epsilon x^3 - x \sim 0$ which produces $x \sim \pm\epsilon^{-1/2}$.

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Example: Solve $\epsilon x^3 - x + 1 = 0$ for small ϵ.

Using $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ as before, one obtains

$$-x_0 + 1 = 0, \quad -x_1 + x_0^3 = 0, \quad -x_2 + 3x_0^2 x_1 = 0, \ldots$$

So that $x_0 = 1$, $x_1 = 1$ and $x_2 = 3$. But this is insufficient since it does not find the other roots!

Notice that ϵx^3 was assumed smaller than 1 above, and dominant balance was between the last two terms. Must this always be the case?

Suppose instead dominant balance between first two terms, so that $\epsilon x^3 - x \sim 0$ which produces $x \sim \pm\epsilon^{-1/2}$.
To find correction term, insert $x \sim \pm\epsilon^{-1/2} + x_1$ where $x_1 = o(\epsilon^{-1/2})$:

$$0 = \epsilon(\pm\epsilon^{-1/2} + x_1)^3 - (\pm\epsilon^{-1/2} + x_1) + 1 = 2x_1 + 1 + o(1),$$

which means $x_1 = -1/2$.

Singular perturbations

Problems which are not regular perturbations are called singular perturbations. In these cases, the form of the expansion cannot always be guessed ahead of time, and each term is found by successive use of the method of dominant balance.

Example: Solve $\epsilon x^3 - x + 1 = 0$ for small ϵ.

Using $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ as before, one obtains

$$-x_0 + 1 = 0, \quad -x_1 + x_0^3 = 0, \quad -x_2 + 3x_0^2 x_1 = 0, \ldots$$

So that $x_0 = 1$, $x_1 = 1$ and $x_2 = 3$. But this is insufficient since it does not find the other roots!

Notice that ϵx^3 was assumed smaller than 1 above, and dominant balance was between the last two terms. Must this always be the case?

Suppose instead dominant balance between first two terms, so that $\epsilon x^3 - x \sim 0$ which produces $x \sim \pm\epsilon^{-1/2}$.
To find correction term, insert $x \sim \pm\epsilon^{-1/2} + x_1$ where $x_1 = o(\epsilon^{-1/2})$:

$$0 = \epsilon(\pm\epsilon^{-1/2} + x_1)^3 - (\pm\epsilon^{-1/2} + x_1) + 1 = 2x_1 + 1 + o(1),$$

which means $x_1 = -1/2$.

Finally, it is possible to have dominant balance between the first and third terms? If that were the case, then $\epsilon x^3 + 1 \sim 0$ or $x = \mathcal{O}(\epsilon^{-1/3})$. But then the second term would actually be bigger than the other two, not smaller.

Dominant balance at lower orders

It can happen that further expansion terms also need to check various cases for dominant balance.

Example: Solve $(1-\epsilon)x^2 - 2x + 1 = 0$ for $\epsilon \to 0^+$.

Dominant balance at lower orders

It can happen that further expansion terms also need to check various cases for dominant balance.

Example: Solve $(1-\epsilon)x^2 - 2x + 1 = 0$ for $\epsilon \to 0^+$.

For the leading order problem, ϵx^2 is always smaller than the other terms, hence

$$x_0^2 - 2x_0 + 1 = 0$$

so that $x_0 = 1$.

Dominant balance at lower orders

It can happen that further expansion terms also need to check various cases for dominant balance.

Example: Solve $(1-\epsilon)x^2 - 2x + 1 = 0$ for $\epsilon \to 0^+$.

For the leading order problem, ϵx^2 is always smaller than the other terms, hence

$$x_0^2 - 2x_0 + 1 = 0$$

so that $x_0 = 1$. For a correction term, set $x \sim 1 + x_1$ with $x_1 = o(1)$, giving

$$\begin{aligned} 0 &\sim (1-\epsilon)(1 + 2x_1 + x_1^2) - 2(1 + x_1) + 1 \\ &= x_1^2 - \epsilon - 2\epsilon x_1 - \epsilon x_1^2. \end{aligned}$$

Dominant balance at lower orders

It can happen that further expansion terms also need to check various cases for dominant balance.

Example: Solve $(1-\epsilon)x^2 - 2x + 1 = 0$ for $\epsilon \to 0^+$.

For the leading order problem, ϵx^2 is always smaller than the other terms, hence

$$x_0^2 - 2x_0 + 1 = 0$$

so that $x_0 = 1$. For a correction term, set $x \sim 1 + x_1$ with $x_1 = o(1)$, giving

$$\begin{aligned} 0 &\sim (1-\epsilon)(1 + 2x_1 + x_1^2) - 2(1 + x_1) + 1 \\ &= x_1^2 - \epsilon - 2\epsilon x_1 - \epsilon x_1^2. \end{aligned}$$

Dominant balance at lower orders

It can happen that further expansion terms also need to check various cases for dominant balance.

Example: Solve $(1-\epsilon)x^2 - 2x + 1 = 0$ for $\epsilon \to 0^+$.

For the leading order problem, ϵx^2 is always smaller than the other terms, hence

$$x_0^2 - 2x_0 + 1 = 0$$

so that $x_0 = 1$. For a correction term, set $x \sim 1 + x_1$ with $x_1 = o(1)$, giving

$$\begin{aligned} 0 &\sim (1-\epsilon)(1 + 2x_1 + x_1^2) - 2(1 + x_1) + 1 \\ &= x_1^2 - \epsilon - 2\epsilon x_1 - \epsilon x_1^2. \end{aligned}$$

Since $x_1 = o(1)$, then $2\epsilon x_1 << \epsilon$ and $\epsilon x_1^2 << \epsilon$, therefore the only two terms that can balance give $0 = x_1^2 - \epsilon$, or $x_1 = \pm\epsilon^{1/2}$. Notice here ϵ must be positive to have any solutions at all.

Transcendental equations

For transcendental equations, order functions may not be powers.

Example: Solve $xe^{-x} = \epsilon$ for small ϵ.

Transcendental equations

For transcendental equations, order functions may not be powers.

Example: Solve $xe^{-x} = \epsilon$ for small ϵ.

First write this as

$$\ln x - x = \ln \epsilon.$$

There are three cases for dominant balance to consider:
Case I: $\ln x - x \sim 0$. Here, there is no solution at all! Even if there were, then $x = \mathcal{O}(1)$ which would mean that the $\ln \epsilon$ term is dominant instead.

Transcendental equations

For transcendental equations, order functions may not be powers.

Example: Solve $xe^{-x} = \epsilon$ for small ϵ.

First write this as

$$\ln x - x = \ln \epsilon.$$

There are three cases for dominant balance to consider:
Case I: $\ln x - x \sim 0$. Here, there is no solution at all! Even if there were, then $x = \mathcal{O}(1)$ which would mean that the $\ln \epsilon$ term is dominant instead.

Case II: $\ln x \sim \ln \epsilon$. Then $x \sim \epsilon$, which is allowed by dominant balance since the $-x$ term is smaller than the other two. In fact, a whole series $x = \epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \ldots$ can be developed. Inserting into the equation and (Taylor) expanding the logarithm,

$$\ln \epsilon + \epsilon a_2 - 2\epsilon^2 a_3 + \ldots - (\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + \ldots) = \ln \epsilon,$$

so that, for example, $\epsilon a_2 = \epsilon$ or $a_2 = 1$.

Transcendental equation example, cont.

Case III: $x \sim \ln(1/\epsilon)$, which also works for dominant balance. Inserting $x = \ln(1/\epsilon) + x_1$ with $x_1 = o(\ln(1/\epsilon))$,

$$0 \sim \ln(\ln(1/\epsilon) + x_1) - x_1 \sim \ln\ln(1/\epsilon) - x_1,$$

so that $x_1 = \ln\ln(1/\epsilon)$. Inserting $x = \ln(1/\epsilon) + \ln\ln(1/\epsilon) + x_2$ produces

$$0 \sim \frac{\ln\ln(1/\epsilon)}{\ln(1/\epsilon)} - x_2.$$

Thus we have $x = \ln(1/\epsilon) + \ln\ln(1/\epsilon) + \frac{\ln\ln(1/\epsilon)}{\ln(1/\epsilon)} + \dots$.

Transcendental equation example, cont.

Case III: $x \sim \ln(1/\epsilon)$, which also works for dominant balance. Inserting $x = \ln(1/\epsilon) + x_1$ with $x_1 = o(\ln(1/\epsilon))$,

$$0 \sim \ln(\ln(1/\epsilon) + x_1) - x_1 \sim \ln\ln(1/\epsilon) - x_1,$$

so that $x_1 = \ln\ln(1/\epsilon)$. Inserting $x = \ln(1/\epsilon) + \ln\ln(1/\epsilon) + x_2$ produces

$$0 \sim \frac{\ln\ln(1/\epsilon)}{\ln(1/\epsilon)} - x_2.$$

Thus we have $x = \ln(1/\epsilon) + \ln\ln(1/\epsilon) + \frac{\ln\ln(1/\epsilon)}{\ln(1/\epsilon)} + \dots$.

Remark: another way to generate an expansion is by the (contractive) mapping

$$x_{n+1} = \ln x_n + \ln(1/\epsilon),$$

which generates the same sequence of approximations.