Asymptotic Methods
Asymptotic series and approximations

Asymptotic sequences and series

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The quantity $f(\epsilon)$ has an asymptotic expansion (or series)

$$f(\epsilon) \sim \sum_{k=1}^{n} a_k \phi_k(\epsilon)$$

with $n \leq \infty$, in the asymptotic sequence $\phi_1(\epsilon), \phi_2(\epsilon), \dots$, provided

$$f(\epsilon) - \sum_{k=1}^{m} a_k \phi_k(\epsilon) = o(\phi_m), \quad \epsilon \to 0$$

for any $1 \leq m \leq n$.

Asymptotic expansions, examples

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Example 2: Often Taylor expansion of terms within an expression lead to an expansion:

$$\begin{aligned}\frac{1}{1-e^{\epsilon}} \sim \frac{1}{1-(1+\epsilon+\epsilon^2/2!+\dots)} &= -\epsilon^{-1}\frac{1}{1+\epsilon/2!+\epsilon^2/3!+\dots} \\ &= -\epsilon^{-1}\left(1-\epsilon/2!-\epsilon^2/3!-\dots+(\epsilon/2!+\epsilon^2/3!+\dots)^2+\dots\right) \\ &= -\epsilon^{-1}+\frac{1}{2}-\frac{1}{12}\epsilon+\dots.\end{aligned}$$

Asymptotic expansions, examples

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Since x/n is small, can expand the logarithm

$$\begin{aligned}\exp[n \ln(1 + x/n)] &\sim exp[x - x^2/(2n) + x^3/(3n^2) - \ldots] \\ &= \exp(x) \exp(-x^2/2n + x^3/(3n^2) + \ldots) \\ &= \exp(x)[1 - x^2/(2n) + x^3/(3n^2) + \mathcal{O}(n^{-4}].\end{aligned}$$

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$$1 - \epsilon = \tanh y = \frac{1 - e^{-2y}}{1 + e^{-2y}} \sim 1 - 2e^{-2y} + 2e^{-4y} + \ldots, \quad y \to \infty.$$

Dominant balance: largest terms in this expression yield $\epsilon \sim 2e^{-2y}$ or $y \sim -\frac{1}{2}\ln(\epsilon/2)$.

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To find next term in expansion, let $y \sim -\frac{1}{2}\ln(\epsilon/2) + y_1$ with $y_1 = o(1)$, and repeat dominant balance argument:

$$\begin{aligned} 1-\epsilon &\sim 1 - 2\exp(\ln(\epsilon/2) - 2y_1) + 2\exp(2\ln(\epsilon/2) - 4y_1) \\ &= 1 - \epsilon\exp(-2y_1) + \frac{\epsilon^2}{2}\exp(-4y_1) \\ &\sim 1 - \epsilon(1 - 2y_1 + \dots) + \frac{\epsilon^2}{2}(1 - 4y_1 + \dots) \end{aligned}$$

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After cancellation, dominant terms give $0 \sim 2\epsilon y_1 - \epsilon^2/2$, so that $y_1 = \epsilon/4$. Further corrections could be obtained using $y \sim -\frac{1}{2}\ln(\epsilon/2) + \epsilon/4 + y_2$.

Remarks

- If the form of the expansion is known in advance, i.e.

$$f(x) \sim a_1\phi_1(\epsilon) + a_2\phi_2(\epsilon) + a_3\phi_3(\epsilon) + \ldots,$$

  it can be substituted into the problem, and terms of each order $\phi_1, \phi_2, \phi_3, \ldots$ can be equated. Typically, all but the leading order term a_1 satisfies a linear equation. This feature makes finding expansions tractable.

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- An infinite asymptotic series does not need to converge. For example, formally computing

$$\int_0^\infty \frac{e^{-t}}{1+\epsilon t} dt \sim \int_0^\infty e^{-t} \left(\sum_{n=0}^{\infty} (-1)^n t^n \epsilon^n \right) dt$$
$$\sim \sum_{n=0}^{\infty} (-1)^n n! \epsilon^n.$$

  leads to series that clearly does not converge unless $\epsilon = 0$, but can be shown to be asymptotic.