Asymptotic Methods

Bifurcation analysis

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For simplicity consider scalar problem with $x_0 = 0 = \lambda_0$, and suppose

$$f(x, \lambda) \sim \lambda x + Ax^n + \dots, \quad x \to 0.$$

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Regarding $\epsilon = \lambda - \lambda_0$ as the small parameter, we seek solutions to the fixed points equation of the form $x \sim x_0 + \epsilon^\alpha x_1 + \dots$, where dominant balance suggests $\lambda x \sim -Ax^n$, so that $\alpha = 1/(n-1)$ and $x_1 = -A^{-1/(n-1)}$.

Higher dimensional example

Consider the system

$$\dot{x} = \mu x + y + \sin x, \quad \dot{y} = x - y.$$

For the fixed point $(0, 0)$, linearization gives $\nabla F = \begin{pmatrix} \mu + 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is singular when $\mu = -2$.

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where $\epsilon = \mu + 2$. Inserting into the fixed point equations and collecting powers of ϵ gives

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} -x_1 + x_1^3/6 \\ 0 \end{pmatrix}.$$

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The first two equations say that $(x_1, y_1)^T$ and $(x_2, y_2)^T$ are in the nullspace of the matrix, therefore $(x_1, y_1)^T = c(1,1)^T$ for some c.

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The first two equations say that $(x_1, y_1)^T$ and $(x_2, y_2)^T$ are in the nullspace of the matrix, therefore $(x_1, y_1)^T = c(1,1)^T$ for some c.

The third equation has a solvability condition, obtained by taking a dot product with the nullspace eigenvector $(1,1)^T$, giving $(-x_1 + x^3/6, 0)^T \cdot (1,1)^T$ or $x_1 = y_1 = \pm\sqrt{6}$.

Therefore, for $\mu < -2$, there is one fixed point, whereas for $\mu > -2$, there are three branches: $(0,0)$ and $\sim \pm\epsilon^{1/2}(\sqrt{6}, \sqrt{6})$.

Nonlinear boundary value problem

Consider

$$u_{xx} + \lambda u + f(u) = 0, \quad u(0) = 0 = u(1), \quad f(u) = Cu^3 + \mathcal{O}(u^4).$$

Clearly $u(x) = 0$ is a solution. Are there other solutions which bifurcate from this as λ is varied?

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Linearization about $u = 0$ gives

$$\mathcal{L}u \equiv u_{xx} + \lambda u = 0, \quad u(0) = 0 = u(1).$$

This problem is singular when λ is an eigenvalue, i.e. $\lambda_0 = (n\pi)^2$. Note the corresponding eigenfunction is $\sin(n\pi x)$.

Nonlinear boundary value problem, cont.

Expanding $u = \epsilon^{1/2} u_1 + \epsilon u_2 + \epsilon^{3/2} u_3 + \ldots$ with $\epsilon = \lambda - \lambda_0$, one obtains $\mathcal{L} u_1 = 0$, $\mathcal{L} u_2 = 0$ and

$$\mathcal{L} u_3 = -u_1 - C u_1^3.$$

It follows that $u_{1,2} = A_{1,2} \sin(n\pi x)$. The equation for u_3 has a solvability condition obtained by multiplying by the nullspace element $\sin(n\pi x)$ and integrating, giving

$$\int A \sin^2(n\pi x) + A^3 C \sin^4(n\pi x)\, dx = 0, \quad \text{or } A = \pm\sqrt{-4/3C}.$$

Nonlinear boundary value problem, cont.

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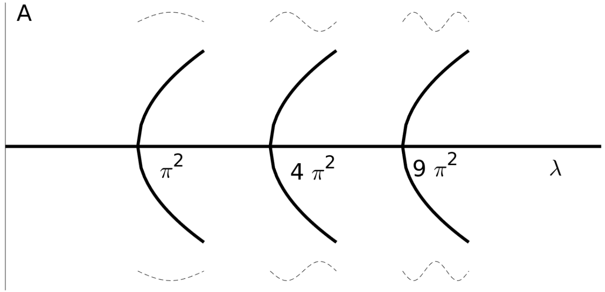
Notice that if $C < 0$, one has a supercritical pitchfork, but if $C > 0$, there are no bifurcations.



An ecological example

Consider a population density $u(x, t)$ of fish, trapped between the shore and shark-infested deep water. Subject to logistic growth and diffusion, this satisfies

$$u_t = Du_{xx} + u(1 - u), \quad u_x(0, t) = 0, \quad u(1, t) = 0.$$

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Steady solutions satisfy a nonlinear BVP

$$u_{xx} + \alpha^2 u(1-u) = 0, \quad u_x(0) = 0, \quad u(1) = 0.$$

Linearizing about $u = 0$, we have the eigenvalue problem

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In this case, bifurcations might be expected when $\alpha = \pi/2 + n\pi$, $n = 0, 1, 2, \ldots$

Ecological example, cont.

Consider $n = 0$ case, and let $\alpha = (\pi/2)(1 + \alpha_1 \epsilon)$, $u \sim \epsilon u_1 + \epsilon^2 u_2 + \ldots$, where $\epsilon \ll 1$. At leading order, one has

$$\mathcal{L} u_1 \equiv u_{1xx} + \frac{\pi^2}{4} u_1 = 0, \quad u_{1x}(0) = 0, \quad u_1(1) = 0.$$

Therefore $u_1 = A \cos(\pi x/2)$.

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At order ϵ^2, the problem is

$$\mathcal{L} u_2 = -\frac{\pi^2}{4}(2\alpha_1 u_1 - u_1^2), \quad u_{2x}(0) = 0, \quad u_2(1) = 0.$$

Solvability is obtained by taking an inner product with $\cos(\pi x/2)$, leading to

$$\frac{\pi^2}{4} \int_0^1 A^2 \cos^3(\pi x/2) - 2\alpha_1 A \cos^2(\pi x/2)\, dx = 0, \quad \text{or } \alpha = 4A/(3\pi)$$

Ecological example, cont.

What about stability of both $u = 0$ and the bifurcating solution? Returning to the time dependent problem,

$$u_t = Du_{xx} + u(1-u), \quad u_x(0,t) = 0, \quad u(1,t) = 0,$$

first linearize about $u = 0$, giving

$$v_t = Dv_{xx} + v, \quad v_x(0,t) = 0, \quad v(1,t) = 0.$$

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With $v = e^{\lambda t}\phi(x)$, get eigenvalue problem

$$\phi_{xx} + \alpha^2(1 - \lambda)\phi = 0, \quad \phi_x(0) = 0, \quad \phi(1) = 0.$$

The eigenvalues compute to be

$$\lambda = 1 - \kappa_j^2/\alpha^2, \quad \kappa_j = (2j - 1)\pi/2, \quad j = 1, 2, 3, \ldots$$

Then $u = 0$ is unstable precisely when $\alpha > \pi/2$, i.e. where the bifurcating solution appears.

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What about stability of nontrivial steady states? Let $w(x)$ be any steady solution; with $u(x,t) = w(x) + v(x,t)$, linearization produces

$$v_t = Dv_{xx} + (1-2w)v, \quad v_x(0,t) = 0, \quad v(1,t) = 0.$$

Again with $v = e^{\lambda t}\phi(x)$, get eigenvalue problem

$$\phi_{xx} + \alpha^2((1-2w(x)) - \lambda)\phi = 0, \quad \phi_x(0) = 0, \quad \phi(1) = 0.$$

Even if we knew an exact expression for $w(x)$, this would be a hard problem since the coefficients are not constant. We can, however, approximate this problem *near* the bifurcation value $\alpha \approx \pi/2$.

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Let $w = \epsilon\cos(\pi x/2) + \ldots$ and $\alpha = \pi/2(1 + 4\epsilon/(3\pi) + \ldots)$. Inserting this into the eigenvalue problem and approximating at leading order, one gets

$$\phi_{xx} + \frac{\pi^2}{4}(1-\lambda)\phi = 0, \quad \phi_x(0) = 0, \quad \phi(1) = 0,$$

which produces eigenvalues $\lambda_j = 1 - (2j-1)^2$ for $j = 1, 2, 3, \ldots$. All of these are negative except $\lambda_1 = 0$.

Ecological example, cont.

To sort out the true size of the leading eigenvalue, we should have tried $\lambda = \epsilon\lambda_1$, and expanded further as $\phi = \cos(\pi x/2) + \epsilon\phi_1$. The ϵ size terms in the eigenvalue problem give

$$\phi_{1xx} + \frac{\pi^2}{4}\phi_1 = \frac{\pi^2}{4}(2\cos(\pi x/2) + \lambda_1) - \frac{4}{3}\cos(\pi x/2).$$

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This is inhomogeneous, and has a solvability condition which is obtained as before, giving

$$\lambda_1 = -\frac{4}{\pi^2}\frac{(\pi^2/2 - 4/3)\int_0^1 \cos^2(\pi x/2)\,dx}{\int_0^1 \cos(\pi x/2)\,dx} \leq 0.$$

The nonzero solution branch is in fact stable. This situation is an example of a *transcritical bifurcation*.