Asymptotic Methods

Hopf bifurcations and delay differential equations

# Hopf bifurcation and differential delay equations

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Linearization produces $\dot{x} = \nabla F(0, r)x$. Two common scenarios for instability:
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Consider differential delay equation $u_t = -au(t-\tau) +$ nonlinear terms.
Inserting $u = e^{i\sigma t}$ into the linear part, get

$$i\sigma = -a[\cos(\sigma t) - i\sin(\sigma t)]$$

Hopf must occur where $\sigma$ is real, i.e. where $\sigma\tau = \pi/2 + n\pi$ and $\sigma = a\sin(\sigma\tau)$.
First occurrence is $\tau = \pi/(2a)$.

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Hopf bifurcation is at $\tau_0 = \pi/(2a)$, so let $\tau = \tau_0 + \epsilon$, let $s = \epsilon t$ be slow scale, expand

$$u = u_0(x, t, s) + \epsilon u_1(x, t, s) + \dots$$

Note delay term expands

$$u(x, t - \tau, s) = u(x, t - \tau_0, s) - \epsilon u_t(x, t - \tau_0, s) - \epsilon \tau_0 u_s(x, t - \tau_0, s)$$

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Order $\epsilon$ terms give

$$u_{1t} = u_{0xx} - bu_0^3 - u_{0s}(x, t - \tau_0, s) + au_{0t}(x, t - \tau_0, s) + a\tau_0 u_{0s}(x, t - \tau_0, s).$$

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Observe that $u_0(x, t - \tau_0, s) = Ae^{-i\pi/2}e^{iat} = -iAe^{iat}$, and the cubic term expands

$$(Ae^{iat} + \text{c.c.})^3 = |A|^2 Ae^{iat} + \text{c.c.} + \text{non-secular terms}$$

Therefore to avoid resonance, get complex Ginzburg-Landau equation

$$(1 + i\pi/2)A_s = A_{xx} + A(a^2 - 3b|A|^2).$$

# Epidemic model with delayed infection

SIQR model with delayed infection (Wang & Chen, 2016)

$$\dot{S} = \Lambda - \frac{\beta S(t-\tau) I(t-\tau)}{S(t-\tau) + I(t-\tau)} - dS,$$

$$\dot{I} = \frac{\beta S(t-\tau) I(t-\tau)}{S(t-\tau) + I(t-\tau)} - (c_1 + k + d + \mu_1) I,$$

$$\dot{Q} = kI - (c_2 + d + \mu_2) Q,$$

$$\dot{R} = c_1 I + c_2 Q - dR,$$

Equilibrium:

$$S^* = \frac{\Lambda}{\beta + d - (c_1 + k + d + \mu_1)},$$

$$I^* = \frac{[\beta - (c_1 + k + d + \mu_1)]\Lambda}{(c_1 + k + d + \mu_1)[\beta + d - (c_1 + k + d + \mu_1)]}.$$

There are several intervals $[\tau_c, \tau_c^-]$ where Hopf bifurcation exists.

# Normal form via multiple scales

Expand just past onset $\tau = \tau_c + \epsilon$,

$$\mathbf{x}(t) = \varepsilon \mathbf{x}_1 (T_0, T_2) + \varepsilon^2 \mathbf{x}_2 (T_0, T_2) + \varepsilon^3 \mathbf{x}_3 (T_0, T_2) + O(\varepsilon^4),$$

where $T_n = \epsilon^n t$. Leading order solution:

$$\mathbf{x}_1 = A(T_2)\, \mathbf{u} e^{i\omega T_0} + \overline{A}(T_2)\, \overline{\mathbf{u}} e^{-i\omega T_0}$$

After a long calculation, get normal form

$$\dot{a} = a R_1 + \frac{1}{4} R_{11\bar{1}} a^3,$$

$$\dot{\theta} = I_1 + \frac{1}{4} I_{11\bar{1}} a^2,$$

where $A = a \exp(i\theta)$. If $R_{111} < 0$ supercritical, i.e. exists stable oscillations