Asymptotic Methods

Applications to Fourier integrals in PDEs

Example: large time behavior of diffusion

Consider the diffusion equation

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The Fourier transform

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satisfies the initial value problem $\hat{u}_t = -k^2 \hat{u}$ with $\hat{u}(k,0) = \hat{f}(k)$, and the PDE solution is given by the inverse transform

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For $t \gg 0$, this is a Laplace integral with Laplace point $k = 0$. For x fixed, the behavior is

$$u(x,t) \sim \frac{\hat{f}(0)}{2\pi} \left(\frac{\pi}{t}\right)^{1/2} = \frac{\hat{f}(0)}{\sqrt{4\pi t}}.$$

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This approximation is not uniform in x, however. Instead, only expanding $\hat{f}$ at Laplace point gives

$$u(x,t) \sim \frac{\hat{f}(0)}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t + ikx} dk = \frac{\hat{f}(0)}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

In other words, the diffusion equation “forgets” the initial data!

Example: large time behavior of a dispersive waves

Consider the Schrödinger equation

$$iu_t + u_{xx} = 0, \quad u(x,0) = f(x),$$

whose Fourier solution is

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Set $x/t = v$ and consider $t \to \infty$:

$$u(x,t) = \frac{1}{2\pi} \int_{\infty}^{\infty} \hat{f}(k) e^{i(kv - k^2)t}\, dk.$$

This is a stationary phase type integral, with stationary phase point $k^* = v/2$,

$$u(x,t) \sim \frac{\hat{f}(v/2)}{2\pi} \int_{\infty}^{\infty} \exp\left(i(v^2 t/4 - (k - k^*)^2) \right) dk = \frac{\hat{f}(v/2)}{\sqrt{4\pi t}} e^{iv^t/4}.$$

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Suppose that $\hat{f}$ describes a "wave packet" with maximum at k_{max}. How can we run along side the wave so that the amplitude is maximum?
The correct choice should be $v = 2k_{max}$; this is the group velocity.