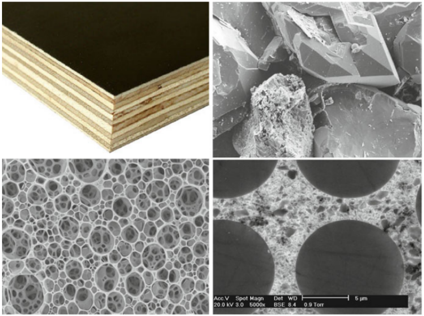
Asymptotic Methods

Homogenization

Elimination of small scales

A wide variety of models are multiscale in space (composite materials, fluid flow through porous media, local social dynamics which influence global behaviors), time (molecular versus cellular processes in biology, mechanical vibrations), or both.



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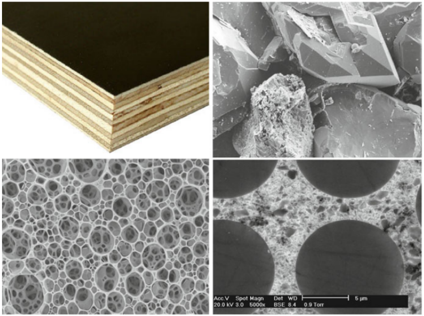

The goal of homogenization is to replace a multiscale model with an "effective" set of equations which do not have the same multiscale structure.

An example of this are the amplitude/envelope equations we derived in pattern-forming equations.

Example

Consider boundary value problem with rapidly varying coefficients

$$[D(x)u']' = f(x), \quad u(0) = a, u(1) = b,$$

where $D = D(x, x/\epsilon)$ with $0 < D_{min} < D < D_{max}$ and $\epsilon \ll 1$.

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Natural to use a multiscale expansion
$u(x,y) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \ldots$ where $y = x/\epsilon$ is the fine-scale variable. This leads to

$$(\partial_y + \epsilon\partial_x)[D(x,y)(\partial_y + \epsilon\partial_x)u] = \epsilon^2 f(x).$$

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Leading order problem is $(D(x,y)u_{0y})_y = 0$, which may be integrated to

$$u_0 = c_1(x) + c_0(x)\int_0^y \frac{ds}{D(x,s)}.$$

Note $\int_0^y ds/D(x,s) \geq y/D_{max}$, so that $u_0 = \mathcal{O}(\epsilon^{-1})$ unless $c_0(x) = 0$. It follows that $u_0 = u_0(x)$.

Example, cont.

At $\mathcal{O}(\epsilon)$, get $(Du_{1y}) = -(u_{0x})D_y$, so that

$$u_1 = b_1(x) + b_0(x)\int_0^y \frac{ds}{D(x,s)} - yu_{0x}.$$

Need to ensure u_1 is not $\mathcal{O}(\epsilon)$, so need

$$\lim_{y\to\infty} \frac{1}{y}\left[b_0(x)\int_0^y \frac{ds}{D(x,s)} - yu_{0x}\right] = 0,$$

or $u_{0x} = b_0(x)\langle D^{-1}\rangle$ where $\langle D^{-1}\rangle = \lim_{y\to\infty} \frac{1}{y}\int_0^y \frac{ds}{D(x,s)}$.

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At $\mathcal{O}(\epsilon^2)$, $(Du_{2y})_y = f(x) - b_0'(x) - (Du_{1x})_y$, so that

$$u_2 = d_1(x) + d_0(x)\int_0^y \frac{ds}{D(x,s)} - \int_0^y u_{1x}(x,s)ds + [f(x) - b_0'(x)]\int_0^y \frac{s\,ds}{D(x,s)}.$$

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The last integral is $\mathcal{O}(y^2)$, and therefore the expansion becomes disordered unless $f(x) = b_0'(x)$.

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Plugging into above expression, get homogenized equation

$$(\overline{D}u_{0x})_x = f(x).$$

where $\overline{D} = 1/\langle D^{-1}\rangle$.

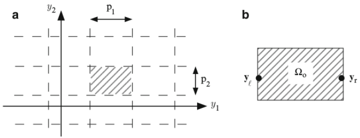
For example, if D is p-periodic in y, $\overline{D}(x) = (p^{-1}\int_0^p ds/D(x,s))^{-1}$, i.e. harmonic average of D.

Multiple dimensions

Consider

$$\nabla \cdot (D \nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^n,$$

Where $D(x, y) = D(x, y + p)$ with $y = x/\epsilon$.



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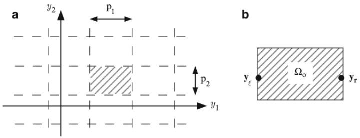

Assume $u(x, y)$ is periodic in y, and expand as before to obtain

$$(\nabla_y + \epsilon\nabla_x) \cdot [D(x, y)(\nabla_y + \epsilon\nabla_x)u] = \epsilon^2 f(x).$$

Leading order is $\nabla_y \cdot (D\nabla_y u_0) = 0$. We can show:

Lemma: If u_0 is periodic in y then $u_0 = u_0(x)$.

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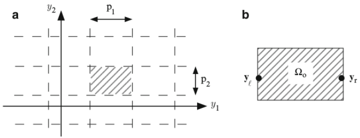

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Lemma: If u_0 is periodic in y then $u_0 = u_0(x)$.
Proof: multiply by u_0 integrate over cell B:

$$0 = \int_B u_0 \nabla_y \cdot (D \nabla u_0) dy = -\int D |\nabla_y u_0|^2 dy,$$

and therefore $\nabla_y u_0 = 0$.

Multiple dimensions,cont.

At $\mathcal{O}(\epsilon)$, have $\nabla_y \cdot (D\nabla_y u_1) = -\nabla_y D \cdot \nabla_x u_0$. This equation is linear and inhomogeneous, and by lemma complimentary solution is a function of x alone. The particular solution is a linear combination $u_1 = a(y) \cdot \nabla_x u_0$ where each component of the vector a solves a cell problem on B,

$$\nabla_y \cdot (D\nabla_y a_i) = -\partial y_i D,$$

subject to periodic boundary conditions.

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At $\mathcal{O}(\epsilon^2)$, have

$$\nabla_y \cdot (D \nabla_y u_2) = -\nabla_y \cdot (D \nabla_x u_1) - \nabla_x \cdot (D \nabla_y u_1) - \nabla_x \cdot (D \nabla_x u_0) + f(x).$$

This inhomogeneous equation has a solvability condition using the inner product $\langle u, v \rangle = \int_B uv \, dy$. Note that the nullspace is simply constants in y.

Multiple dimensions,cont.

Write $|B|^{-1}\langle u, 1\rangle = \langle u\rangle_B$. Thus solvability condition is just averaging the equation over the cell. We have

$$\langle f(x)\rangle_B = f(x), \quad \langle \nabla_x \cdot (D\nabla_x u_0)\rangle_B = \nabla_x \cdot (\langle D\rangle_B \nabla_x u_0), \quad \nabla_y \cdot (\langle D\rangle_B \nabla_x u_1) = 0,$$

where the last equality uses the divergence theorem.

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In addition,

$$\langle D\nabla_y u_1\rangle_B = \langle D\nabla_y (a \cdot \nabla u_0)\rangle_B = \sum_j \langle D\frac{\partial a_j}{\partial y_i}\rangle_B \frac{\partial u_0}{\partial x_j}.$$

It follows that

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(T_{ij}\frac{\partial u_0}{\partial x_j}\right) = f(x),$$

which may be written compactly at $\nabla\cdot(T\nabla u_0) = f(x)$ where T is a tensor with components

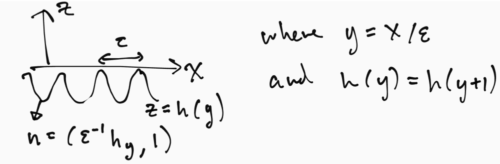
$$T_{ij} = \langle D\rangle_B \delta_{ij} + \langle D\partial y_i a_j\rangle_B.$$

Example: Diffusion through rough interface

Consider

$$\nabla\cdot(\sigma\nabla u) + au = f(x,z), \quad [u] = 0, \quad [\sigma\nabla u \cdot n] = 0.$$

The diffusion constant is $\sigma = \sigma_+$ for $z > h(x/\epsilon)$ and σ_- otherwise.



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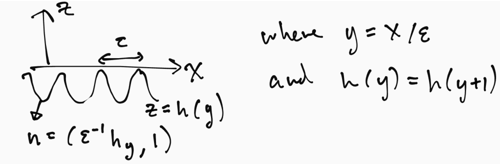

Use multiple scale expansion $u(x,y,z) = u_0(y,x,z) + u_1(y,x,z) + \dots$ where

$$(\sigma u_y)_y + \epsilon(\sigma u_x)_y + \epsilon(\sigma u_y)_x + \epsilon^2[(\sigma u_x)_x + (\sigma u_z)_z + au - f] = 0,$$

and

$$\nabla u \cdot n = ((\partial_x + \epsilon^{-1}\partial_y)u, u_z) \cdot n = \epsilon^{-2} u_y h_y + \epsilon^{-1} u_x h_y - u_z.$$

We suppose that $u_0, u_1, \dots$ are periodic in y.

Diffusion through a rough interface, cont.

Leading order solves $(\sigma u_{0y})_y = 0$ and $[u_0] = 0 = [\sigma u_{0y}]$ on interface, so that

$$u_0 = c_1(x,z)\int_0^y \frac{ds}{\sigma(y)} + c_2(x,z).$$

Periodicity implies $c_1 = 0$.

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At $\mathcal{O}(\epsilon)$, have $(\sigma u_{1y} + \sigma u_{0x})_y = 0$ whose interface conditions are $[u_1] = 0 = [\sigma(u_{1y} + u_{0x})]$. Let $v(x,y)$ be defined by $vu_{0x} = u_1 + yu_{0x}$ so that $v_y u_{0x} = u_{1y} + u_{0x}$.

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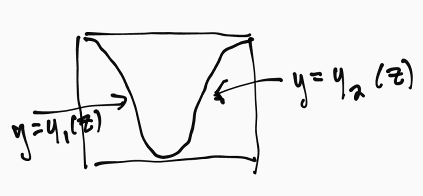
$$(\sigma v_y)_y = \frac{1}{u_{0x}}[\sigma(u_{1y} + u_{0x})]_y = 0, \quad [v] = 0, \quad [\sigma v_y] = 0.$$

This means that σv_y is independent of y, so let it equal $\sigma_e(z)$. Note also that $v = u_1/u_{0x} + y$, so by periodicity $v(x, y+1, z) = v(x,y,z) + 1$.

Diffusion through a rough interface, cont.

Let $y = y_{1,2}(z)$ define each part of the interface.



Integrate equation for v from y_1 to y,

$$v = v(x, y_1, z) + \begin{cases} (\sigma_e/\sigma_+)(y - y_1), & y_1 < y < y_2, \\ (\sigma_e/\sigma_+)(y_2 - y_1) + (\sigma_e/\sigma_-)(y - y_2), & y_2 < y < y_1 + 1. \end{cases}$$

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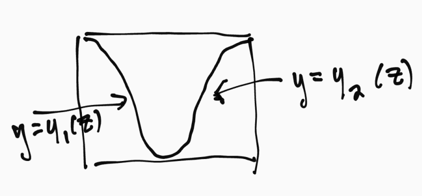

Integrate equation for v from y_1 to y,

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Using $v(x, y + 1, z) = v(x, y, z) + 1$, have

$$v(x, y_1, z) + (\sigma_e/\sigma_+)(y_2 - y_1) + (\sigma_e/\sigma_-)(y_1 + 1 - y_2) = v(x, y_1, z) + 1$$

which means $\sigma_e(z) = [(y_2(z) - y_1(z))/\sigma_+ + (y_1(z) - y_2(z) + 1)/\sigma_-]^{-1}$.
Observe σ_e interpolates between $\sigma_\pm$.

Diffusion through a rough interface, cont.

Order ϵ^2 has

$$(\sigma u_{2y} + \sigma u_{1x})_y = -(\sigma u_{1y} + \sigma u_{0x})_x - (\sigma u_0)_{zz} - a u_0 - f.$$

and $[u_2] = 0 = [\sigma(u_{2y} h_y + u_{1x} h_y - u_{0z})]$.

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Use $u_{1y} + u_{0x} = v_y u_{0x}$, and average over fine scale by integrating y from 0 to 1:

$$\sigma_e u_{0xx} + \langle \sigma \rangle u_{0zz} + a u_0 - f = -\int_0^1 \sigma (u_{2y} + \sigma u_{1x})_y dy.$$

where $\langle \sigma \rangle = \int_0^1 \sigma dy = \sigma_+ (y_2(z) - y_1(z)) + \sigma_- (1 - y_2(z) + y_1(z))$.

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where $\langle\sigma\rangle = \int_0^1 \sigma dy = \sigma_+(y_2(z) - y_1(z)) + \sigma_-(1 - y_2(z) + y_1(z))$.

The integral has a discontinuous integrand, and evaluates to

$$[\sigma(u_{2y} + u_{1x})]_{y=y_1} + [\sigma(u_{2y} + u_{1x})]_{y=y_2} = \frac{[\sigma u_{0z}]_{y_1}}{h_y(y_1)} + \frac{[\sigma u_{0z}]_{y_2}}{h_y(y_2)}$$
$$= (\sigma_+ - \sigma_-)[1/h_y(y_2) - 1/h_y(y_1)]u_{0z} = \langle\sigma\rangle_z.$$

Finally, the homogenized equation can be written as

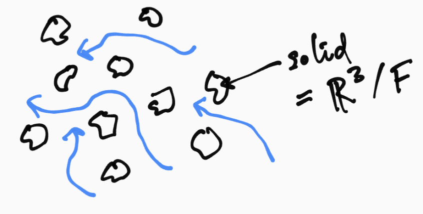
$$(\langle\sigma\rangle u_{0z})_z + \sigma_e u_{0xx} + au_0 = f.$$

Flow through porous media

Consider a random but prescribed characteristic function $\omega : \mathbb{R}^3 \to \mathbb{R}$,

$$\omega(x) = \begin{cases} 1, & x \text{ in fluid region } F, \\ 0, & x \text{ in solid region } S \end{cases}$$

which varies on a fine scale $y = x/\epsilon$.



Flow field $v(x)$ and pressure satisfy the the Stokes equations

$$\begin{aligned} \mu\Delta v &= \nabla p, \quad x \in F, \\ \nabla\cdot v &= 0, \quad x \in F, \\ v &= 0, \quad x \in \partial F. \end{aligned}$$

Flow through porous media, cont.

We use multiscale expansion $v = \epsilon^2 v_0(x, y) + \epsilon^3 v_1(x, y) + \dots$ and $p = p_0(x) + \epsilon p_1(x, y)$ and obtain

$$\begin{aligned} \mu\Delta_y v_0 &= \nabla_y p_1 + \nabla_x p_0 \\ \nabla_y \cdot v_0 &= 0 \\ \nabla_x \cdot v_0 + \nabla_y \cdot v_1 &= 0. \end{aligned}$$

Homogenization is obtained by averaging over ensemble. Define $U(x) = \langle v_0 \rangle(x)$. Third equation averages to $\nabla_x \cdot U + \langle \nabla_y \cdot v_1 \rangle = 0$.

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Since $\langle \nabla_y \cdot v_1 \rangle$ is independent of y, letting B_R be a ball we can write

$$\begin{aligned} \langle \nabla_y \cdot v_1 \rangle &= \frac{1}{|B_R|} \int_{B_R} \langle \nabla_y \cdot v_1 \rangle dy = \frac{1}{|B_R|} \left\langle \int_{B_R} \nabla_y \cdot v_1 \, dy \right\rangle \\ &= \left\langle \frac{1}{|B_R|} \int_{\partial F} v_1 \cdot n \, dy \right\rangle + \left\langle \frac{1}{|B_R|} \int_{\partial B_R} v_1 \cdot n \, dy \right\rangle . \end{aligned}$$

Taking $R \to \infty$ gives zero, so we conclude $\nabla \cdot U = 0$.

Flow through porous media

Let matrix W and vector Π solve

$$\begin{aligned} \Delta W &= \nabla\Pi - I, \quad x \in F, \\ \nabla\cdot W &= 0, \quad x \in F, \\ W &= 0, \quad x \in \partial F. \end{aligned}$$

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Then $v_0 = -(1/\mu) W \cdot p_0$ and $p_1 = \Pi \cdot \nabla p_0$. Averaging the former gives

$$U(x) = -\frac{1}{\mu} \langle W \rangle \cdot \nabla p_0.$$

The quantity $\langle W \rangle$ is called the permeability tensor. Together with the incompressibility condition $\nabla \cdot U = 0$, we obtain Darcy's law.