Asymptotic Methods

Approximations of integrals: introduction

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The simplest case is where the integrand possess an expansion $f(t;x) \sim \sum_{k=0}^{\infty} f_k(t)\phi_k(x)$ which is uniform in $t \in [a,b]$. This means that for any $C > 0$, x can be chosen small (or large) enough so that

$$\left| f(t;x) - \sum_{k=0}^{N} f_k(t)\phi_k(x) \right| = C\phi_k(x), \quad \text{for each } N = 1, 2, 3, \ldots$$

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For example, as $x \to 0$,

$$\int_0^1 \frac{\sin(tx)}{t}\, dt \sim \int_0^1 \frac{tx - t^3x^3/6 + t^5x^5/120 - \ldots}{t}\, dt = x - x^3/18 + x^5/600 - \ldots$$

where term-by-term integration was used.

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Expanding

$$\frac{1}{1 + xt^2} \sim 1 - xt^2 + x^2 t^4 - \dots$$

produces an asymptotic expansion which is not uniform in t; moreover term-by-term integration would be impossible. In fact, the integral can be computed exactly to give $\pi/(2\sqrt{x})$.

Example: the Stieltjes integral

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For $x \to 0$, one has

$$\frac{e^{-t}}{1+xt} \sim e^{-t}(1 - xt + x^2t^2 - \ldots),$$

which is, in fact, a uniform expansion in x (exercise). Then we compute

$$I(x) \sim \sum_{n=0} x^n \int_0^\infty (-1)^n e^{-t} t^n \, dt \sim \sum_{n=0}^{\infty} (-1)^n (n!) x^n.$$

This sum is certainly not convergent unless $x = 0$.

Example: the Stieltjes integral, cont.

What about $x \to \infty$? Write

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Trick: split integral from 0 to x and x to ∞. Then

$$\int_0^x \frac{e^{-w/x} dw}{1 + w} = \int_0^x \frac{1 - w/x + w^2/(2x^2) + w^3/(6w^2) + \dots}{1 + w} dw = \ln(x) + \mathcal{O}(1).$$

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The other integral is

$$\int_x^\infty \frac{e^{-w/x} dw}{1+w} \leq \frac{1}{x} \int_x^\infty e^{-w/x} dw = 1/e.$$

We therefore have the first term $I(x) \sim \ln(x)/x$.

Another example

The so-called exponential integral is

$$E_1 = \int_x^\infty \frac{e^{-t}}{t} dt.$$

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The inspiration here is to differentiate and expand,

$$E_1'(x) = -\frac{e^{-x}}{x} = -\frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{n!}.$$

Then

$$E_1 = -\ln x + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n \cdot n!} + C.$$

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Add $\ln x$ to both sides and take $x \to 0$:

$$C = \lim_{x \to 0^+} \left(\int_x^\infty \frac{e^{-t}}{t} dt + \ln x \right) = \int_0^\infty e^{-t} \ln t \, dt \equiv -\gamma,$$

Here integration by parts was used, and γ is the famous EulerMascheroni constant $\gamma = 0.57721\ldots$

A difficult example

Look for behavior as $x \to \infty$ for

$$I(x) = \int_0^1 \frac{e^x - e^{xt}}{1-t} dt = e^x \int_0^1 \frac{1 - e^{-xu}}{u} \, du.$$

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Simply ignoring the term e^{-xu} leads to a divergent integral. In fact, e^{-xu} is not small if $xu \ll 1$, which suggests splitting the integral as

$$\int_0^{1/x} \frac{1 - e^{-xu}}{u}\, du + \int_{1/x}^1 \frac{1 - e^{-xu}}{u}\, du.$$

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The second integral can be written

$$\ln x - \int_1^x \frac{e^{-w}}{w}\,dw = \ln x - E_1(1) + E_1(x).$$

So we are left looking for large x behavior of the exponential integral, which will be investigated in the next topic.