Asymptotic Methods

Approximations of integrals: integration by parts

Series from integration by parts

A useful method of generating an asymptotic expansion of an integral is integration by parts.
For example,

$$\int_0^x f(t)dt = f(t)x - \int_0^x f(t)tdt = f(t)x - f'(t)x^2/2 + \dots$$

gives a series in powers of x, appropriate for $x \to 0$.
Integration by parts the other way leads to series in negative powers of x, which would be useful for $x \to \infty$.

The exponential integral revisited

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Integration by parts gives the recursion formula

$$E_m = -t^{-m}e^{-t}\Big|_x^\infty - m\int_x^\infty e^{-t}/t^{m+1}\,dt = x^{-m}e^{-x} - mE_{m+1}.$$

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Therefore repeated integration by parts produces

$$E_1(x) = \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + \frac{2e^{-x}}{x^3} + \ldots + (-1)^{n-1}(n-1)!\frac{e^{-x}}{x^n} + (-1)^{n-1}(n!)\int_x^\infty e^{-t}t^{n+1} dt.$$

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Note that

$$\int_x^\infty e^{-t}t^{n+1}dt \le x^{-(n+1)}\int_x^\infty e^{-t}dt = e^{-x}x^{-(n+1)},$$

which is in fact $o(e^{-x}x^{-n})$.

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For $x \to \infty$, IBP produces

$$\begin{aligned}
\int_x^\infty \frac{1}{t}[\sin(t^2)/2]'dt &= -\frac{\sin x^2}{2x} + \frac{1}{2}\int_x^\infty \frac{1}{t^3}[-\cos(t^2)/2]'dt \\
&= -\frac{\sin x^2}{2x} + \frac{\cos(x^2)}{4x^3} - \frac{1\cdot 3}{4}\int_x^\infty \frac{1}{t^5}[\sin(t^2)/2]'dt \\
&= -\frac{\sin x^2}{2x} + \frac{\cos(x^2)}{4x^3} - \frac{1\cdot 3}{8}\sin(x^2) - \frac{1\cdot 3\cdot 5}{8}\int_x^\infty \frac{1}{t^7}[-\cos(t^2)/2]'dt \\
&= -\frac{\sin x^2}{2x} + \frac{\cos(x^2)}{4x^3} - \frac{1\cdot 3}{8x^5}\sin(x^2) + \frac{1\cdot 3\cdot 5}{16x^7}\cos(x^2) + \ldots
\end{aligned}$$