Asymptotic Methods

Approximations of integrals: Watson's Lemma and Laplace's method

Laplace's method

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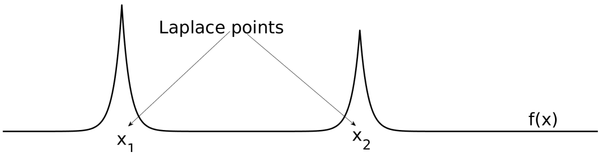

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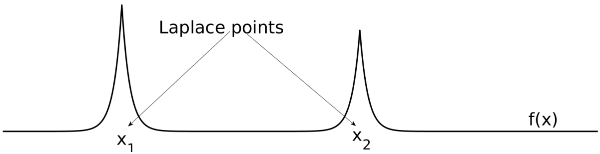

Take as an example

$$I(x) = \int_0^1 \frac{e^{-xt}}{1+t}\,dt, \quad x \to \infty$$

Expanding $1/(1+t) = 1 - t + t^2 - t^3 - \ldots$ leads to

$$I(x) \sim \sum_{n=0}^{\infty} \int_0^1 (-t)^n e^{-xt}\,dt.$$

Unfortunately, term by term integration leads to something quite messy, because of the fact that the upper bound is finite.

Laplace's method, cont.

Writing $\int_0^1 = \int_0^\infty - \int_1^\infty$,

$$\left| \int_1^\infty (-t)^n e^{-xt} dt \right| \leq \int_1^\infty e^{-xt} dt = \mathcal{O}(e^{-x}/x).$$

This contribution is exponentially small compared to the other term:

$$\sum_{n=0}^\infty \int_0^\infty (-t)^n e^{-xt} dt = \sum_{n=0}^\infty (-1)^n n! x^{-n-1} dt.$$

In other words, essentially all the integral's value came from around $t = 0$; this observation leads to Laplace's method.

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Watson's lemma

Suppose $f(t) \sim t^{\alpha}(a_0 t^{\beta} + a_1 t^{2\beta} + a_2 t^{3\beta} + \ldots)$ for $t \to 0$, then

$$\int_0^b f(t)e^{xt}dt \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}, \quad x \to \infty$$

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(I) $\phi(x)$ has a unique maximum at $x = a$. Then

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(II) $\phi(x)$ has a unique maximum at $x = c$, $a < c < b$. Then

$$I \sim \int_{-\infty}^{\infty} f(a)e^{x(\phi(a)+\phi''(c)(t-c)^2/2)}\, dt = \frac{\sqrt{2\pi} f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}.$$

More generally, heuristic idea is to expand both $f()$ and ϕ "enough" so that a

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The Laplace point here is $t = 0$, and $\phi(t) = -\sin^4 t \sim -t^4$ has a fourth-order maximum there. Also, $\ln(1+t) \sim t$ for $t \to 0$. Therefore

$$I(x) \sim \int_0^{\infty} t \exp(-xt^4) \, dt = \frac{1}{4x^{1/2}} \int_0^{\infty} s^{-1/2} e^{-s} ds = \frac{\Gamma(1/2)}{4x^{1/2}}.$$

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To put in needed form, let $v = cosh(t) - 1$, so that $v \sim \frac{1}{2}t^2$ for small t. To obtain a whole series for t as a function of v, let $t = (2v)^{1/2} + c_1 v_1 + c_2 v^{3/2} + \ldots$, and substitute into expansion of $\cosh(t) - 1$:

$$v = \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots = v + \sqrt{2}c_1 v^{3/2} + v^2(\sqrt{2}c_2 + c_1^2/6 + 1/6) + \ldots$$

Comparing terms on either side, it follows $c_1 = 0$ and $c_2 = -1/(6\sqrt{2})$.

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Finally, after the change of variables in the integral, the expansions are inserted and Watson's lemma applies:

$$e^{-x}\int_0^\infty e^{-xv}[1 + m^2t^2/2! + \ldots](dt/dv)dv$$

$$= e^{-x}\int_0^\infty e^{-xv}\left(\sqrt{2}v^{-1/2}/2 + v^{1/2}[\sqrt{2}m^2/2 - 1/(4\sqrt{2})] + \ldots\right)dv$$

$$= e^{-x}\frac{\sqrt{\pi}}{2x}\left[1 + \frac{1}{2}(m^2 - 1/4)\frac{1}{x} + \ldots\right].$$

Moving maxima

For an integral like

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \int_0^\infty e^{x \ln t - t} / t \, dt,$$

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The Laplace point is at $s = 1$, and represents a quadratic maximum of $\phi(s) = \ln s - s$. Thus

$$\Gamma(x) \sim x^x \int_{-\infty}^\infty e^{-x(1-(s-1)^2/s)} (1) ds = x^x e^{-x} \sqrt{\frac{2\pi}{x}}.$$

This is known as Stirling's approximation.

An exception

Consider the integral

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One possible way forward is a change of variables $z = xt$,

$$I(x) = \frac{1}{x} \int_0^{x/e} \frac{e^{-z}}{\ln z - \ln x} dz.$$

Can't expand in geometric series unless $|\ln z| < |\ln x|$, so split integral $\int_0^{z_c} + \int_{z_c}^{x/e}$, where cutoff must be chosen so $|\ln z_c| < |\ln x|$, e.g. $z_c = x^{-1/2}$.

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$$\frac{1}{x} \int_0^{x^{-1/2}} \frac{e^{-z}}{\ln z - \ln x} dz \leq \frac{1}{x} \int_0^{x^{-1/2}} \frac{e^{-z}}{\ln x^{-1/2} - \ln x} dz = \frac{2}{x^{3/2}} \ln x.$$

and

$$\int_{x^{-1/2}}^{x/e} \frac{e^{-z}}{\ln z - \ln x} dz = -\frac{1}{x \ln x} \sum c_k (\ln x)^{-k}, \quad c_k = \int_0^{\infty} e^{-z} (\ln z)^k dz.$$

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The integral may be written

$$P(A) = \frac{L^{1/2}}{Z} \int_S e^{-Ls^2} ds,$$

which is of Laplace form. The Laplace point is $s^* = min(S) > 0$, so that

$$P(A) \sim \frac{L^{1/2}}{Z} \int_S e^{-L[(s^*)^2 + 2s^*(s-s^*)]} ds = \frac{e^{-L(s^*)^2}}{2ZL^{1/2}s^*}.$$

This is an example of a large deviation result.