Asymptotic Methods

Approximations of integrals: Watson’s Lemma and Laplace’s method
Laplace’s method

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Take as an example

\[ I(x) = \int_{0}^{1} \frac{e^{-xt}}{1 + t} \, dt, \quad x \to \infty \]

Expanding \( 1/(1 + t) = 1 - t + t^2 - t^3 - \ldots \) leads to

\[ I(x) \sim \sum_{n=0}^{\infty} \int_{0}^{1} (-t)^n e^{-xt} \, dt. \]

Unfortunately, term by term integration leads to something quite messy, because of the fact that the upper bound is finite.
Writing \( \int_0^1 = \int_0^\infty - \int_1^\infty \),

\[
\left| \int_1^\infty (-t)^n e^{-xt} \, dt \right| \leq \int_1^\infty e^{-xt} \, dt = O(e^{-x}/x).
\]

This contribution is exponentially small compared to the other term:

\[
\sum_{n=0}^\infty \int_0^\infty (-t)^n e^{-xt} \, dt = \sum_{n=0}^\infty (-1)^n n! x^{-n-1} \, dt.
\]

In other words, essentially all the integral’s value came from around \( t = 0 \); this observation leads to Laplace’s method.
Watson’s lemma

The previous intuition can be formalized in a well-known result

**Watson’s lemma**

Suppose \( f(t) \sim t^\alpha (a_0 t^\beta + a_1 t^{2\beta} + a_2 t^{3\beta} + \ldots) \) for \( t \to 0 \), then

\[
\int_0^b f(t)e^{xt} \, dt \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}, \quad x \to \infty
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Many Laplace integrals are not exactly of this form, but the same intuition applies. Consider

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l(x) = \int_a^b e^{x\phi(t)} f(t) dt.
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Two common cases are:
(I) $\phi(x)$ has a unique maximum at $x = a$. Then

$$l \sim \int_a^\infty f(a) e^{x(\phi(a) + \phi'(a)(t-a))} dt = - \frac{f(a)e^{x\phi(a)}}{x\phi'(a)}$$
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Many Laplace integrals are not exactly of this form, but the same intuition applies. Consider

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Two common cases are:

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(II) \( \phi(x) \) has a unique maximum at \( x = c, a < c < b \). Then

\[
I \sim \int_{-\infty}^\infty f(a) e^{x(\phi(a) + \phi''(c)(t-c)^2/2)} dt = \frac{\sqrt{2\pi} f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}.
\]

More generally, heuristic idea is to expand both \( f() \) and \( \phi ""enough"" so that a
Examples

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\[ I(x) \sim \int_{0}^{\infty} t \exp(-xt^4) \, dt = \frac{1}{4x^{1/2}} \int_{0}^{\infty} s^{-1/2} e^{-s} \, ds = \frac{\Gamma(1/2)}{4x^{1/2}}. \]
Exact transformation to Watson

If more than a leading order approximation is desired, an exact change of variables allows Watson’s lemma to be used.

Example: modified Bessel functions have the representation

\[ K_m(x) = \int_0^\infty e^{-x \cosh t} \cosh(mt) \, dt \]

To put in needed form, let \( v = \cosh(t) - 1 \), so that \( v \sim \frac{1}{2} t^2 \) for small \( t \). To obtain a whole series for \( t \) as a function of \( v \), let \( t = (2v)\frac{1}{2} + c_1v + c_2v^3/2 + \ldots \), and substitute into expansion of \( \cosh^{-1}(t) \):

\[ v = \frac{t^2}{2} + \frac{t^4}{4!} + \ldots = v + \sqrt{2}c_1v^3/2 + v^2(\sqrt{2}c_2 + c_2/6 + 1/6) + \ldots \]

Comparing terms on either side, it follows \( c_1 = 0 \) and \( c_2 = -1/(6\sqrt{2}) \).

Finally, after the change of variables in the integral, the expansions are inserted and Watson’s lemma applies:

\[ e^{-x} \int_0^\infty e^{-xv} \left[ 1 + m^2 t^2/2! + \ldots \right] \left( \frac{dt}{dv} \right) \, dv = e^{-x} \int_0^\infty e^{-xv} \left( \sqrt{2}v - 1/2 + v^2(\sqrt{2}m^2/2 - 1/4) + \ldots \right) \, dv = e^{-x} \sqrt{\pi/2} x \left[ 1 + 1/2 (m^2 - 1/4) x + \ldots \right] \]
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\]
For an integral like

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt = \int_0^\infty e^{x \ln t - t} / t \, dt, \]

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Letting \( t = sx \),

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The Laplace point is at \( s = 1 \), and represents a quadratic maximum of \( \phi(s) = \ln s - s \). Thus

\[ \Gamma(x) \sim x^x \int_{-\infty}^\infty e^{-x(1-(s-1)^2)/s}(1) ds = x^x e^{-x} \sqrt{\frac{2\pi}{x}}. \]

This is known as Stirling’s approximation.
Consider the integral

\[ I(x) = \int_{0}^{1/e} \frac{e^{-xt}}{\ln t} \, dt, \quad x \to \infty. \]

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\[ I(x) = \frac{1}{x} \int_{0}^{x/e} \frac{e^{-z}}{\ln z - \ln x} \, dz. \]

Can’t expand in geometric series unless \( |\ln z| < |\ln x| \), so split integral \( \int_{0}^{z_c} + \int_{z_c}^{x/e} \), where cutoff must be chosen so \( |\ln z_c| < |\ln x| \), e.g. \( z_c = x^{-1/2} \).
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where cutoff must be chosen so \( |\ln z_c| < |\ln x| \), e.g. \( z_c = x^{-1/2} \). Then

\[ \frac{1}{x} \int_0^{x^{-1/2}} \frac{e^{-z}}{\ln z - \ln x} \, dz \leq \frac{1}{x} \int_0^{x^{-1/2}} \frac{e^{-z}}{\ln z - \ln x} \, dz = \frac{2}{x^{3/2}} \ln x. \]

and

\[ \int_{x^{-1/2}}^{x/e} \frac{e^{-z}}{\ln z - \ln x} \, dz = -\frac{1}{x \ln x} \sum c_k (\ln x)^{-k}, \quad c_k = \int_0^\infty e^{-z}(\ln z)^k \, dz. \]
Laplace integrals in probability

What is the probability of a rare event, and what does it look like?
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Suppose $A = L^{1/2} S$ where $S$ is some fixed set; would like behavior for $L \to \infty$. 

The integral may be written

$$P(A) = \frac{1}{Z} \int_S e^{-Ls^2} ds,$$

which is of Laplace form. The Laplace point is $s^* = \min(S) > 0$, so that

$$P(A) \sim \frac{1}{Z} \int_S e^{-L(s^* + (s - s^*)^2 / 2)} ds = e^{-L(s^*)^2 / 2}.$$ 

This is an example of a large deviation result.
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