Asymptotic Methods

Local analysis of ordinary differential equations

Regular and irregular singular points

Consider linear differential equation

$$y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \ldots + p_0(x)y = 0.$$

We would like to know how this behaves in the neighborhood of a point x_0. Thus we seek an expansion in the small quantity $|x - x_0| \ll 1$.

Regular and irregular singular points

Consider linear differential equation

$$y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \ldots + p_0(x)y = 0.$$

We would like to know how this behaves in the neighborhood of a point x_0. Thus we seek an expansion in the small quantity $|x - x_0| \ll 1$.

Three cases to consider:

1. If $p_j(x_0)$ are analytic near x_0, x_0 is called an ordinary point. A theorem due to Fuchs says that $y(x)$ is also analytic, and therefore has a solution as a power series in $x - x_0$.

Regular and irregular singular points

Consider linear differential equation

$$y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \ldots + p_0(x)y = 0.$$

We would like to know how this behaves in the neighborhood of a point x_0. Thus we seek an expansion in the small quantity $|x - x_0| \ll 1$.

Three cases to consider:

1. If $p_j(x_0)$ are analytic near x_0, x_0 is called an ordinary point. A theorem due to Fuchs says that $y(x)$ is also analytic, and therefore has a solution as a power series in $x - x_0$.
2. If $p_{n-k}(x)(x - x_0)^k$ are analytic near x_0, x_0 is called a regular singular point.

Regular and irregular singular points

Consider linear differential equation

$$y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \ldots + p_0(x)y = 0.$$

We would like to know how this behaves in the neighborhood of a point x_0. Thus we seek an expansion in the small quantity $|x - x_0| \ll 1$.

Three cases to consider:

1. If $p_j(x_0)$ are analytic near x_0, x_0 is called an ordinary point. A theorem due to Fuchs says that $y(x)$ is also analytic, and therefore has a solution as a power series in $x - x_0$.
2. If $p_{n-k}(x)(x - x_0)^k$ are analytic near x_0, x_0 is called a regular singular point. Reason for this definition: consider special case $x_0 = 0$ and $p_{n-k}(x)x^k = 1$; this gives an Euler type equation

$$y^{(n)}(x) + \frac{1}{x}y^{(n-1)}(x) + \ldots = 0.$$

   The solutions here are generally powers of x, although degeneracy leads to necessary logarithmic corrections. This leads to a general recipe for expansions, known as the method of Frobenius.

Regular and irregular singular points

Consider linear differential equation

$$y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \ldots + p_0(x)y = 0.$$

We would like to know how this behaves in the neighborhood of a point x_0. Thus we seek an expansion in the small quantity $|x - x_0| \ll 1$.

Three cases to consider:

1. If $p_j(x_0)$ are analytic near x_0, x_0 is called an ordinary point. A theorem due to Fuchs says that $y(x)$ is also analytic, and therefore has a solution as a power series in $x - x_0$.
2. If $p_{n-k}(x)(x - x_0)^k$ are analytic near x_0, x_0 is called a regular singular point. Reason for this definition: consider special case $x_0 = 0$ and $p_{n-k}(x)x^k = 1$; this gives an Euler type equation

   $$y^{(n)}(x) + \frac{1}{x}y^{(n-1)}(x) + \ldots = 0.$$

   The solutions here are generally powers of x, although degeneracy leads to necessary logarithmic corrections. This leads to a general recipe for expansions, known as the method of Frobenius.
3. Everything else gives irregular singular points. Unlike RSP, there is no official algorithm to find an expansion.

We can also classify $x_0 = \infty$ by the substitution $t = 1/x$ and consider $t = 0$ instead.

Expansions around irregular singular points

Whereas the theory for expansions around regular and regular singular points is known, irregular singular points must be investigated on a case-by-case basis.

Expansions around irregular singular points

Whereas the theory for expansions around regular and regular singular points is known, irregular singular points must be investigated on a case-by-case basis.

For the simple example of a ISP at $x = 0$, consider $y' = yx^{-p}$. It has the solution

$$y = C \exp\left(\frac{x^{1-p}}{1-p}\right).$$

The behavior for small x is not a power, but a function of an exponential.

Expansions around irregular singular points

Whereas the theory for expansions around regular and regular singular points is known, irregular singular points must be investigated on a case-by-case basis.

For the simple example of a ISP at $x = 0$, consider $y' = yx^{-p}$. It has the solution

$$y = C \exp\left(\frac{x^{1-p}}{1-p}\right).$$

The behavior for small x is not a power, but a function of an exponential.

A general ansatz that works for ISP is $y \sim \exp(S(x))$. After substitution, a series for $S(x)$ which (hopefully) provides a good approximation for the original problem.

Expansions around irregular singular points

Whereas the theory for expansions around regular and regular singular points is known, irregular singular points must be investigated on a case-by-case basis.

For the simple example of a ISP at $x = 0$, consider $y' = yx^{-p}$. It has the solution

$$y = C \exp\left(\frac{x^{1-p}}{1-p}\right).$$

The behavior for small x is not a power, but a function of an exponential.

A general ansatz that works for ISP is $y \sim \exp(S(x))$. After substitution, a series for $S(x)$ which (hopefully) provides a good approximation for the original problem.

Simple lemma: If $S(x) = \sum_{j=1}^{N} S_j(x) + o(1)$ for $x \to 0$, then

$$\exp(S(x)) \sim \exp \sum_{j=1}^{N} S_j(x).$$

In other words, we need to expand S to enough terms so the error is small in absolute terms.

ISP example

Consider $x^3 y'' = y$ for $x \to 0^+$. Setting $y = \exp(S(x))$, get $S'' + (S')^2 = x^{-3}$.

ISP example

Consider $x^3 y'' = y$ for $x \to 0^+$. Setting $y = \exp(S(x))$, get $S'' + (S')^2 = x^{-3}$.
There are 3 cases to consider for dominant balance:
I. $S'' + (S')^2 \sim 0$. Integrating leads to $S \sim \ln x$, but then $x^{-3} \gg S''$.
II. $S'' \sim x^{-3}$, leads to $S \sim x^{-1}$, but then $(S')^2 \sim x^{-4}$.
III. $(S')^2 = x^{-3}$, which can be integrated to $S = \pm 2x^{-1/2}$.

ISP example

Consider $x^3 y'' = y$ for $x \to 0^+$. Setting $y = \exp(S(x))$, get $S'' + (S')^2 = x^{-3}$. There are 3 cases to consider for dominant balance:
I. $S'' + (S')^2 \sim 0$. Integrating leads to $S \sim \ln x$, but then $x^{-3} \gg S''$.
II. $S'' \sim x^{-3}$, leads to $S \sim x^{-1}$, but then $(S')^2 \sim x^{-4}$.
III. $(S')^2 = x^{-3}$, which can be integrated to $S = \pm 2x^{-1/2}$.

To improve the approximation, let $S = 2x^{-1/2} + C(x)$, where

$$\frac{3}{2}x^{-5/2} + C'' - 2x^{-3/2}C' + (C')^2 = 0, \quad C = o(x^{-1/2}).$$

If C is like a power or logarithm, then $C'' \ll x^{-3/2}C'$. Since $C = o(x^{-1/2})$, it also holds that $(C')^2 \ll x^{-3/2}C'$. Thus dominant balance implies $\frac{3}{2}x^{-5/2} - 2x^{-3/2}C' \sim 0$, which integrates to $C(x) \sim \frac{3}{4}\ln x$.

ISP example

Consider $x^3 y'' = y$ for $x \to 0^+$. Setting $y = \exp(S(x))$, get $S'' + (S')^2 = x^{-3}$.
There are 3 cases to consider for dominant balance:
I. $S'' + (S')^2 \sim 0$. Integrating leads to $S \sim \ln x$, but then $x^{-3} \gg S''$.
II. $S'' \sim x^{-3}$, leads to $S \sim x^{-1}$, but then $(S')^2 \sim x^{-4}$.
III. $(S')^2 = x^{-3}$, which can be integrated to $S = \pm 2x^{-1/2}$.

To improve the approximation, let $S = 2x^{-1/2} + C(x)$, where

$$\frac{3}{2}x^{-5/2} + C'' - 2x^{-3/2}C' + (C')^2 = 0, \quad C = o(x^{-1/2}).$$

If C is like a power or logarithm, then $C'' \ll x^{-3/2}C'$. Since $C = o(x^{-1/2})$, it also holds that $(C')^2 \ll x^{-3/2}C'$. Thus dominant balance implies $\frac{3}{2}x^{-5/2} - 2x^{-3/2}C' \sim 0$, which integrates to $C(x) \sim \frac{3}{4}\ln x$.

Further expansion of $S(x)$ leads to terms which are $\mathcal{O}(1)$, so by the lemma we now have a good approximation. To make even more progress, try $y \sim Cx^{3/4}\exp(2x^{-1/2})[1 + w(x)]$, with $w = o(1)$, where

$$w'' + (\frac{3}{2} - 2x^{-3/2})w' - \frac{3}{16}x^{-2} - \frac{3}{16}wx^{-2} \sim 0.$$

Dominant balance gives $-2x^{-3/2}w' \sim \frac{3}{16}wx^{-2}$, so $w = \frac{3}{16}x^{1/2}$.

ISP example

Consider $x^3 y'' = y$ for $x \to 0^+$. Setting $y = \exp(S(x))$, get $S'' + (S')^2 = x^{-3}$.
There are 3 cases to consider for dominant balance:
I. $S'' + (S')^2 \sim 0$. Integrating leads to $S \sim \ln x$, but then $x^{-3} \gg S''$.
II. $S'' \sim x^{-3}$, leads to $S \sim x^{-1}$, but then $(S')^2 \sim x^{-4}$.
III. $(S')^2 = x^{-3}$, which can be integrated to $S = \pm 2x^{-1/2}$.

To improve the approximation, let $S = 2x^{-1/2} + C(x)$, where

$$\frac{3}{2}x^{-5/2} + C'' - 2x^{-3/2}C' + (C')^2 = 0, \quad C = o(x^{-1/2}).$$

If C is like a power or logarithm, then $C'' \ll x^{-3/2}C'$. Since $C = o(x^{-1/2})$, it also holds that $(C')^2 \ll x^{-3/2}C'$. Thus dominant balance implies $\frac{3}{2}x^{-5/2} - 2x^{-3/2}C' \sim 0$, which integrates to $C(x) \sim \frac{3}{4}\ln x$.

Further expansion of $S(x)$ leads to terms which are $\mathcal{O}(1)$, so by the lemma we now have a good approximation. To make even more progress, try $y \sim Cx^{3/4}\exp(2x^{-1/2})[1 + w(x)]$, with $w = o(1)$, where

$$w'' + (\frac{3}{2} - 2x^{-3/2})w' - \frac{3}{16}x^{-2} - \frac{3}{16}wx^{-2} \sim 0.$$

Dominant balance gives $-2x^{-3/2}w' \sim \frac{3}{16}wx^{-2}$, so $w = \frac{3}{16}x^{1/2}$.
In fact, after a lot of labor, one can find a whole series for $w(x)$

$$y \sim Cx^{3/4}\exp(2x^{-1/2})\sum_{n=0}^{\infty}\frac{\Gamma(n-1)\Gamma(n+3/2)}{\pi 4^n n!}\left(\frac{x}{2}\right)^{n/2}.$$

Some nonlinear examples

Consider the nonlinear equation

$$y' = \left(\frac{1}{x} - \frac{1}{y^2}\right) y, \quad x \to \infty.$$

Dominant balance is still a useful tool, but there may be many viable cases.

Some nonlinear examples

Consider the nonlinear equation

$$y' = \left(\frac{1}{x} - \frac{1}{y^2}\right) y, \quad x \to \infty.$$

Dominant balance is still a useful tool, but there may be many viable cases.

First suppose $y' \sim y/x$, which gives $y \sim Cx$. This is allowed, since $1/y^2 << 1/x$.

Some nonlinear examples

Consider the nonlinear equation

$$y' = \left(\frac{1}{x} - \frac{1}{y^2}\right) y, \quad x \to \infty.$$

Dominant balance is still a useful tool, but there may be many viable cases.

First suppose $y' \sim y/x$, which gives $y \sim Cx$. This is allowed, since $1/y^2 << 1/x$.

Now try $y' \sim -1/y$. This integrates to $y = \sqrt{2(C - x)}$, bad for large x.

Some nonlinear examples

Consider the nonlinear equation

$$y' = \left(\frac{1}{x} - \frac{1}{y^2}\right) y, \quad x \to \infty.$$

Dominant balance is still a useful tool, but there may be many viable cases.

First suppose $y' \sim y/x$, which gives $y \sim Cx$. This is allowed, since $1/y^2 << 1/x$.

Now try $y' \sim -1/y$. This integrates to $y = \sqrt{2(C - x)}$, bad for large x.

Try $1/x \sim 1/y^2$. This means $y \sim x^{1/2}$, so y' is same size as other terms.

Some nonlinear examples

Consider the nonlinear equation

$$y' = \left(\frac{1}{x} - \frac{1}{y^2}\right) y, \quad x \to \infty.$$

Dominant balance is still a useful tool, but there may be many viable cases.

First suppose $y' \sim y/x$, which gives $y \sim Cx$. This is allowed, since $1/y^2 << 1/x$.

Now try $y' \sim -1/y$. This integrates to $y = \sqrt{2(C-x)}$, bad for large x.

Try $1/x \sim 1/y^2$. This means $y \sim x^{1/2}$, so y' is same size as other terms. We should be looking instead for 3-term balance. Substituting $y = Cx^{\alpha}$ leads to $\alpha = 1/2$ and $C = \sqrt{2}$. This is an exact solution!

Some nonlinear examples, cont.

Consider $y^2 y''' = -1/3$ for $x \to \infty$. Try $y \sim ax^2 + bx + c + w(x)$.

Some nonlinear examples, cont.

Consider $y^2 y''' = -1/3$ for $x \to \infty$. Try $y \sim ax^2 + bx + c + w(x)$.

If $a \neq 0$, dominant terms are $a^2 x^4 w''' \sim -1/3$, so $w \sim 1/(18a^2 x)$. But are there other solutions which don't grow quadratically?

Some nonlinear examples, cont.

Consider $y^2y''' = -1/3$ for $x \to \infty$. Try $y \sim ax^2 + bx + c + w(x)$.

If $a \neq 0$, dominant terms are $a^2x^4w''' \sim -1/3$, so $w \sim 1/(18a^2x)$. But are there other solutions which don't grow quadratically?

Try $y \sim Ax^r$, giving $A^2x^{3r-3} = -1/3$ so one would need $r = 1$. But this fails!

Some nonlinear examples, cont.

Consider $y^2 y''' = -1/3$ for $x \to \infty$. Try $y \sim ax^2 + bx + c + w(x)$.

If $a \neq 0$, dominant terms are $a^2 x^4 w''' \sim -1/3$, so $w \sim 1/(18a^2 x)$. But are there other solutions which don't grow quadratically?

Try $y \sim Ax^r$, giving $A^2 x^{3r-3} = -1/3$ so one would need $r = 1$. But this fails!

Often when an ansatz predicts a solution which does not work, the remedy is to include logarithms. Here, try $y \sim Ax(\ln x)^\alpha$; with

$$y''' \sim A\alpha x^{-2}(\ln x)^{\alpha-1} + \mathcal{O}(x^{-2}(\ln x)^{\alpha-2}).$$

Substitution into the equation gives $-A^3\alpha(\ln x)^{3\alpha-1} \sim -1/3$, so that $\alpha = 1/3$ and $A = 1$.

Some nonlinear examples, cont.

Want behavior as $x \to 0$ in $y'''y' = 2(y'')^2 + y$.

Some nonlinear examples, cont.

Want behavior as $x \to 0$ in $y'''y' = 2(y'')^2 + y$.

Each combination of dominant balance can be tried, but they lead to something non-integrable. A different heuristic is to take $y \sim x^\alpha$; in this case the y term looks subdominant.

Some nonlinear examples, cont.

Want behavior as $x \to 0$ in $y'''y' = 2(y'')^2 + y$.

Each combination of dominant balance can be tried, but they lead to something non-integrable. A different heuristic is to take $y \sim x^\alpha$; in this case the y term looks subdominant.

Inserting $y \sim x^\alpha$ into $y'''y' \sim 2(y'')^2$ leads to $\alpha = 0$ or 1, but these do not work as leading order solutions! Try instead $y \sim (\ln x)^\beta$:

$$y' \sim \beta x^{-1}(\ln x)^{\beta-1}, \quad y'' \sim -\beta x^{-2}(\ln x)^{\beta-1} + \beta(\beta-1)x^{-2}(\ln x)^{\beta-2},$$

$$y''' \sim 2\beta x^{-3}(\ln x)^{\beta-1} - 2\beta(\beta-1)x^{-3}(\ln x)^{\beta-2} + \beta(\beta-1)(\beta-2)x^{-3}(\ln x)^{\beta-3}.$$

Some nonlinear examples, cont.

Want behavior as $x \to 0$ in $y'''y' = 2(y'')^2 + y$.

Each combination of dominant balance can be tried, but they lead to something non-integrable. A different heuristic is to take $y \sim x^\alpha$; in this case the y term looks subdominant.

Inserting $y \sim x^\alpha$ into $y'''y' \sim 2(y'')^2$ leads to $\alpha = 0$ or 1, but these do not work as leading order solutions! Try instead $y \sim (\ln x)^\beta$:

$$y' \sim \beta x^{-1}(\ln x)^{\beta-1}, \quad y'' \sim -\beta x^{-2}(\ln x)^{\beta-1} + \beta(\beta-1)x^{-2}(\ln x)^{\beta-2},$$

$$y''' \sim 2\beta x^{-3}(\ln x)^{\beta-1} - 2\beta(\beta-1)x^{-3}(\ln x)^{\beta-2} + \beta(\beta-1)(\beta-2)x^{-3}(\ln x)^{\beta-3}.$$

Keeping only leading order terms in $y'''y' \sim 2(y'')^2$ gives $2\beta^2 x^{-4}(\ln x)^{2\beta-2} = 2\beta^2 x^{-4}(\ln x)^{2\beta-2}$, which does not select β.

Some nonlinear examples, cont.

Want behavior as $x \to 0$ in $y'''y' = 2(y'')^2 + y$.

Each combination of dominant balance can be tried, but they lead to something non-integrable. A different heuristic is to take $y \sim x^\alpha$; in this case the y term looks subdominant.

Inserting $y \sim x^\alpha$ into $y'''y' \sim 2(y'')^2$ leads to $\alpha = 0$ or 1, but these do not work as leading order solutions! Try instead $y \sim (\ln x)^\beta$:

$$y' \sim \beta x^{-1}(\ln x)^{\beta-1}, \quad y'' \sim -\beta x^{-2}(\ln x)^{\beta-1} + \beta(\beta-1)x^{-2}(\ln x)^{\beta-2},$$
$$y''' \sim 2\beta x^{-3}(\ln x)^{\beta-1} - 2\beta(\beta-1)x^{-3}(\ln x)^{\beta-2} + \beta(\beta-1)(\beta-2)x^{-3}(\ln x)^{\beta-3}.$$

Keeping only leading order terms in $y'''y' \sim 2(y'')^2$ gives $2\beta^2 x^{-4}(\ln x)^{2\beta-2} = 2\beta^2 x^{-4}(\ln x)^{2\beta-2}$, which does not select β.

Going to the next order gives
$-2\beta^2(\beta-1)x^{-4}(\ln x)^{2\beta-2} = -4\beta^2(\beta-1)x^{-4}(\ln x)^{2\beta-3}$.
Therefore β must be $= 1$.

Some nonlinear examples, cont.

Want small x behavior in forced Painleve equation $y'' = y^2 + A/x^4$.

Some nonlinear examples, cont.

Want small x behavior in forced Painleve equation $y'' = y^2 + A/x^4$.

Trying $y \sim Bx^\alpha$ leads to $\alpha = -2$ and B solves $B^2 - 6B + A = 0$. Provided $A < 9$, there are two values of $B = B_\pm$. But which one is relevant? And if $A > 9$, what happens?

Some nonlinear examples, cont.

Want small x behavior in forced Painleve equation $y'' = y^2 + A/x^4$.

Trying $y \sim Bx^\alpha$ leads to $\alpha = -2$ and B solves $B^2 - 6B + A = 0$. Provided $A < 9$, there are two values of $B = B_\pm$. But which one is relevant? And if $A > 9$, what happens?

Transform variables $y = w/x^2$, $t = -\ln x$ giving

$$w_{tt} + 5w_t + 6w = w^2 + A.$$

Notice $x \to 0$ is the same at $t \to \infty$.

Some nonlinear examples, cont.

Want small x behavior in forced Painleve equation $y'' = y^2 + A/x^4$.

Trying $y \sim Bx^\alpha$ leads to $\alpha = -2$ and B solves $B^2 - 6B + A = 0$. Provided $A < 9$, there are two values of $B = B_\pm$. But which one is relevant? And if $A > 9$, what happens?

Transform variables $y = w/x^2$, $t = -\ln x$ giving

$$w_{tt} + 5w_t + 6w = w^2 + A.$$

Notice $x \to 0$ is the same at $t \to \infty$.

Phase plane reveals smaller $w = B_-$ is stable. Thus some solutions of the original equation have the behavior $y = B_- x^{-2}$.

Some nonlinear examples, cont.

Want small x behavior in forced Painleve equation $y'' = y^2 + A/x^4$.

Trying $y \sim Bx^\alpha$ leads to $\alpha = -2$ and B solves $B^2 - 6B + A = 0$. Provided $A < 9$, there are two values of $B = B_\pm$. But which one is relevant? And if $A > 9$, what happens?

Transform variables $y = w/x^2$, $t = -\ln x$ giving

$$w_{tt} + 5w_t + 6w = w^2 + A.$$

Notice $x \to 0$ is the same at $t \to \infty$.

Phase plane reveals smaller $w = B_-$ is stable. Thus some solutions of the original equation have the behavior $y = B_- x^{-2}$.

On the other hand, some solutions (and all if $A > 9$) have blow up where $w_{tt} \sim w^2$, which integrates to $w \sim (C - \sqrt{2/3}t)^{-2}$. **These solutions never reach $t = \infty$ or $x = 0$!**