Asymptotic Methods

Matched asymptotic expansions
Differential equations often give rise to expansions which are not uniformly valid throughout the domain. To remedy this, different expansions are sought in different subdomains, and they are connected by a systematic process known as ”matching”.

\[ \begin{align*}
\epsilon y'' + (1 + \epsilon)y' + y &= 0, \\
y(0) &= 0, \\
y(1) &= 1,
\end{align*} \]

whose exact solution happens to be
\[ y(x) = \frac{e^{-x} - e^{-x}/\epsilon}{e^{-1} - e^{-1}/\epsilon}. \]

Naively setting \( \epsilon = 0 \) in the equation, we have
\[ y' + y = 0. \]

But how can both boundary conditions be satisfied?

Comparing to the exact solution, the root of the problem is clear:
\[ y'' = O(1/\epsilon^2) \quad \text{if} \quad x \approx 0, \]

therefore dominant balance is different there.

For \( x = O(1) \), the outer solution \( y \sim e^{-1} - x \) can be obtained by using the right hand boundary condition.
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Consider the boundary value problem \( \epsilon y'' + (1 + \epsilon)y' + y = 0 \), with \( y(0) = 0 \) and \( y(1) = 1 \), whose exact solution happens to be

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Comparing to the exact solution, the root of the problem is clear: \( y'' = \mathcal{O}(1/\epsilon^2) \) if \( x \approx 0 \), therefore dominant balance is different there.

For \( x = \mathcal{O}(1) \), the outer solution \( y \sim e^{1-x} \) can be obtained by using the right hand boundary condition.
To resolve the behavior for small $x$, use different independent variable $X = x/\epsilon$, giving

$$y_{xx} + (1 + \epsilon)y_x + \epsilon y = 0.$$ 

The leading order problem is therefore $y_{xx} + y_x = 0$, and with the boundary condition $y(0) = 0$ one has the inner solution $y = c(e^{-x} - 1)$. 
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If we assume both approximations are valid in some overlapping region where $x \ll 1$ and $X \gg 1$, the behaviors

$$y \sim \begin{cases} 
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We can form a uniformly valid approximation by adding the two expansions and subtracting off the common behavior, giving the composite expansion
\[
y \sim e^{1-x} - e^{1-x}/\epsilon.
\]
The method of matched asymptotic expansions

Issues still to be addressed:

1. What is the scaling of the boundary layer? (dominant balance)
2. Where is the boundary layer?
3. Is there a systematic way to match expansions?
Suppose that the boundary layer at $x = 0$ uses an inner variable $X = \epsilon x$, and let $x_\eta = x/\eta(\epsilon)$ with $\epsilon \ll \eta(\epsilon) \ll 1$ be some intermediate scale.
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- The expansion \( y_{\text{outer}}(x_\eta) \sim y_0 + \ldots \) is valid for \( x \gg \eta_1(\epsilon) \) where \( \eta_1(\epsilon) \gg \eta(\epsilon) \),

- The expansion \( y_{\text{inner}}(x_\eta) \sim Y_0 + \ldots \) is valid for \( x \ll \eta_2(\epsilon) \) where \( \eta_2(\epsilon) \ll \eta(\epsilon) \).

Then both expansions in \( x_\eta \) are identical ("Kaplan’s" method).

An often easier way to approach this is the rule of van Dyke: write the outer expansion in terms of the inner variables and vice-versa, and expand in \( \epsilon \). These must agree when written in terms of a common independent variable.
Suppose that the outer expansion is given by $\sum_{n=0}^{\infty} \epsilon^n f_n(x)$ and the inner (using scaled variable $X = x/\epsilon$) by $\sum_{m=0}^{\infty} \epsilon^m F_m(X)$. 

Comparing series, it follows that $a_{mk} = \sum_{n+k=m} f(n(0))/k!$, so that 

- $F_0(X) \sim f_0(0)$, $X \to \infty$, 
- $F_1(X) \sim f_1(0) + f_0'(0)X$, $X \to \infty$, 
- $F_2(X) \sim f_2(0) + f_1'(0)X + \frac{1}{2} f_0''(0)X^2$, $X \to \infty$. 

Illustration of van Dyke matching process for simple power series

Suppose that the outer expansion is given by $\sum_{n=0}^{\infty} \epsilon^n f_n(x)$ and the inner (using scaled variable $X = x/\epsilon$) by $\sum_{m=0}^{\infty} \epsilon^m F_m(X)$.

Writing the outer expansion in terms of the inner variable and (re)expanding using Taylor series gives

$$\sum_{n=0}^{\infty} \epsilon^n f_n(\epsilon X) \sim \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \epsilon^{n+k} f_n^{(k)}(0) X^k / k!$$

Similarly, writing the inner expansion in terms of the outer variable

$$\sum_{m=0}^{\infty} \epsilon^m F_m(x/\epsilon) \sim \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_{mk} \epsilon^{m-k} x^k = \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_{mk} \epsilon^m X^k,$$

where polynomial behavior $F_m \sim \sum_{k=0}^{m} a_{mk} X^k$ is assumed.
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Comparing series, it follows that $a_{mk} = \sum_{n+k=m} f_n^{(k)}(0)/k!$, so that

$$F_0(X) \sim f_0(0), \quad X \to \infty,$$

$$F_1(X) \sim f_1(0) + f_0'(0)X, \quad X \to \infty,$$

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Expanding \( y = y_0 + \epsilon y_1 + \ldots \) leads to

\[ (1 + x)y_0' + y_0 = 0, \quad y_0(1) = 1, \quad (1 + x)y_1' + y_1 = -y_0'', \quad y_1(1) = 0. \]

This produces \( y_0 = 2/(1 + x) \) and \( y_1 = 2/(1 + x)^3 - 1/(2(1 + x)). \)
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Letting \( X = x/\epsilon, \ y = Y(X) \), and expanding \( Y = Y_0 + \epsilon Y_1 + \ldots \),

\[ Y_0'' + Y_0' = 0, \quad Y_0(0) = 1, \quad Y_1'' + Y_1' = -Y_0 - XY_0', \quad Y_1(0) = 0, \]

so \( Y_0 = 1 + A_0(e^{-X} - 1) \) and \( Y_1 = A_1(e^{-X} - 1) + A_0(X - \frac{1}{2}X^2e^{-X}) - X \).
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Writing the outer expansion in terms of \( X \) gives
\[ y \sim 2 + \epsilon(-2X + 3/2) + O(\epsilon^2). \]
The inner expansion in terms of \( x \) gives
\[ Y \sim 1 - A_0 + \epsilon(-A_1 + A_0x/\epsilon - x/\epsilon) + O(\epsilon^2) = 1 - A_0 + \epsilon(A_0 - 1)X - \epsilon A_1. \]
Consider
\[ \epsilon y'' + (1 + x)y' + y = 0, \quad y(0) = 1, \quad y(1) = 1. \]
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(1 + x)y_0' + y_0 = 0, \quad y_0(1) = 1, \quad (1 + x)y_1' + y_1 = -y''_0, \quad y_1(1) = 0.
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Y''_0 + Y_0' = 0, \quad Y_0(0) = 1, \quad Y''_1 + Y_1' = -Y_0 - XY_0', \quad Y_1(0) = 0,
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\]
Comparing series, one has \( A_0 = 1 \) and \( A_1 = -3/2 \).
More generally, the inner variable has the form $X = x/\epsilon^\alpha$. The requirement of dominant balance of two terms determines $\alpha$. 

Example: $4\epsilon y'' + 6\sqrt{xy}' - 3y = -3, \quad y(0) = 0, \quad y(1) = 3$. The outer solution is immediately $y \sim 1 + 2e^{\sqrt{x} - 1}$. 

Set $X = x/\epsilon^\alpha$ and $y = Y(X)$ so that $4\epsilon Y'' + 6\epsilon^{-\alpha/2}\sqrt{X Y}' - 3Y = -3$. Balance between first two terms gives $\alpha = 2/3$. 

The leading order inner problem is $Y''/Y = -3/2X$, $Y(0) = 0$, whose solution is $Y = C \int_0^X e^{-z} 3/2 \, dz$. 

For matching, note $y \sim 2/e + 1$ as $x \to 0$, and $Y \sim C \int_0^\infty e^{-z} 3/2 \, dz$, $X \to \infty$. Thus $C = (2/e + 1) \int_0^\infty e^{-z} 3/2 \, dz$. 

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Try boundary layer on left, using $X = x/\epsilon$. This leads to $Y'' = Y'$ whose general solution is $Y = c_1 + c_2 e^X$. The exponential behavior will not match!
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For boundary layer on right, use $X = (x + 1)/\epsilon$, which also gives $Y'' = Y'$ and $Y = c_1 + c_2 e^X$ at leading order. But in this case, matching requires behavior as $X \to -\infty$, which is ok. Using $Y(1) = 1$, one has $c_1 + c_2 = 0$. 
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The outer solution must use the left boundary condition, so $y(x) \sim -ax$. Matching requires

$$X \sim c_1, \quad X \to -\infty, \quad y \sim -a, \quad x \to 1,$$

so that $c_1 = a$. 
Consider

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Having a boundary layer on the left means that the outer solution is \( y \sim x - 2 \), which is negative if \( x \approx 0 \).
The inner solution solves \( Y'' \sim YY' \). But matching implies there must be place in the inner region where \( Y' > 0 \), \( Y'' \geq 0 \), which can't happen if \( Y < 0 \).
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A similar argument applies to a right hand boundary layer. The only other possibility is a layer in the interior, say at \( x = x_0 \). This means the outer solution looks like

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Using \( X = (x - x_0)/\epsilon \), get \( Y'' = YY' \) to leading order. Integrating once gives \( Y' = \frac{1}{2} Y^2 - B^2/2 \), which can be integrated again to produce

\[
Y = B \frac{1 - De^{BX}}{1 + De^{BX}}.
\]

Matching requires \( \lim_{X \to \pm \infty} Y = \mp B = \lim_{x \to x_0 \pm} y(x) \), which means that \( B = x_0 + 1 \) and \( -B = x_0 - 2 \). It follows that \( B = 3/2 \) and \( x_0 = 1/2 \).