Asymptotic Methods

Matched asymptotic expansions

Non-uniformity and matched expansions

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For $x = \mathcal{O}(1)$, the outer solution $y \sim e^{1-x}$ can be obtained by using the right hand boundary condition.

Non-uniformity and matched expansions,cont.

To resolve the behavior for small x, use different independent variable $X = x/\epsilon$, giving

$$y_{XX} + (1+\epsilon)y_X + \epsilon y = 0.$$

The leading order problem is therefore $y_{XX} + y_X = 0$, and with the boundary condition $y(0) = 0$ one has the inner solution $y = c(e^{-X} - 1)$.

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If we assume both approximations are valid in some overlapping region where $x \ll 1$ and $X \gg 1$, the behaviors

$$y \sim \begin{cases} c(e^{-X} - 1) \sim -c, & X \to \infty \\ e^{1-x} \sim e, & x \to 0 \end{cases}$$

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We can form a uniformly valid approximation by adding the two expansions and subtracting off the common behavior, giving the composite expansion

$$y \sim e^{1-x} - e^{1-x/\epsilon}.$$

The method of matched asymptotic expansions

Issues still to be addressed:

1. What is the scaling of the boundary layer? (dominant balance)
2. Where is the boundary layer?
3. Is there a systematic way to match expansions?

Kaplan and van Dyke matching processes

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- The expansion $y_{outer}(x_\eta) \sim y_0 + \dots$ is valid for $x \gg \eta_1(\epsilon)$ where $\eta_1(\epsilon) \gg \eta(\epsilon)$,
- The expansion $y_{inner}(x_\eta) \sim Y_0 + \dots$ is valid for $x \ll \eta_2(\epsilon)$ where $\eta_2(\epsilon) \ll \eta(\epsilon)$.

Then both expansions in x_η are identical ("Kaplan's" method).

An often easier way to approach this is the rule of van Dyke: write the outer expansion in terms of the inner variables and vice-versa, and expand in ϵ. These must agree when written in terms of a common independent variable.

Illustration of van Dyke matching process for simple power series

Suppose that the outer expansion is given by $\sum_{n=0}^{\infty} \epsilon^n f_n(x)$ and the inner (using scaled variable $X = x/\epsilon$) by $\sum_{m=0}^{\infty} \epsilon^m F_m(X)$.

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Writing the outer expansion in terms of the inner variable and (re)expanding using Taylor series gives

$$\sum_{n=0}^{\infty} \epsilon^n f_n(\epsilon X) \sim \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \epsilon^{n+k} f_n^{(k)}(0) X^k / k!$$

Similarly, writing the inner expansion in terms of the outer variable

$$\sum_{m=0}^{\infty} \epsilon^m F_m(x/\epsilon) \sim \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_{mk} \epsilon^{m-k} x^k = \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_{mk} \epsilon^m X^k,$$

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Comparing series, it follows that $a_{mk} = \sum_{n+k=m} f_n^{(k)}(0)/k!$, so that

$$F_0(X) \sim f_0(0), \quad X \to \infty,$$
$$F_1(X) \sim f_1(0) + f_0'(0) X, \quad X \to \infty,$$
$$F_2(X) \sim f_2(0) + f_1'(0) X + \frac{1}{2} f_0''(0) X^2, \quad X \to \infty.$$

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$$Y_0'' + Y_0' = 0, \quad Y_0(0) = 1, \qquad Y_1'' + Y_1' = -Y_0 - XY_0', \quad Y_1(0) = 0,$$

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Writing the outer expansion in terms of X gives

$$y \sim 2 + \epsilon(-2X + 3/2) + \mathcal{O}(\epsilon^2).$$

The inner expansion in terms of x gives

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Comparing series, one has $A_0 = 1$ and $A_1 = -3/2$.

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Set $X = x/\epsilon^{\alpha}$ and $y = Y(X)$ so that

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For matching, note $y \sim 2/e + 1$ as $x \to 0$, and

$$Y \sim C \int_0^\infty e^{-z^{3/2}} dz, \quad X \to \infty.$$

Thus $C = (2/e + 1)/ \int_0^\infty e^{-z^{3/2}} dz$.

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For boundary layer on right, use $X = (x+1)/\epsilon$, which also gives $Y'' = Y'$ and $Y = c_1 + c_2 e^X$ at leading order. But in this case, matching requires behavior as $X \to -\infty$, which is ok. Using $Y(1) = 1$, one has $c_1 + c_2 = 0$.

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The outer solution must use the left boundary condition, so $y(x) \sim -ax$. Matching requires

$$X \sim c_1, \quad X \to -\infty, \qquad y \sim -a, \quad x \to 1,$$

so that $c_1 = a$.

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A similar argument applies to a right hand boundary layer. The only other possibility is a layer in the interior, say at $x = x_0$. This means the outer solution looks like

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Using $X = (x - x_0)/\epsilon$, get $Y'' = YY'$ to leading order. Integrating once gives $Y' = \frac{1}{2}Y^2 - B^2/2$, which can be integrated again to produce

$$Y = B\frac{1 - De^{BX}}{1 + De^{BX}}.$$

Matching requires $\lim_{X \to \pm\infty} Y = \mp B = \lim_{x \to x_0^{\pm}} y(x)$, which means that $B = x_0 + 1$ and $-B = x_0 - 2$. It follows that $B = 3/2$ and $x_0 = 1/2$.