Asymptotic Methods

Matched asymptotic expansions: exponentials, logarithms, multiple layers

Example: wide capillary tube

Fluid meniscus $u(r)$ satisfies

$$\frac{\epsilon^2}{r}\left(\frac{ru'}{[1+(u')^2]^{1/2}}\right)' = u, \quad 0 < r < 1, \quad u'(0) = 0, \quad u'(1) = \tan\theta_0.$$

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For region I, expand $u = \nu_0(\epsilon)u_0(R) + \ldots$, where dominant balance requires $R = r/\epsilon$. Provided $\nu_0 \ll \epsilon$, leading order is

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The solution is a constant C times modified Bessel function

$$I_0 = \frac{1}{\pi}\int_0^\pi e^{R\cos t}dt.$$

Matching needs the behavior for large R, easy by Laplace's method:

$$u_0 \sim \frac{C}{\pi}\int_0^\pi e^{R(1-t^2/2!)}dt = \frac{Ce^R}{\sqrt{2\pi R}} = C\epsilon^{1/2}e^{r/\epsilon}/\sqrt{2\pi}.$$

Capillary tube, cont.

For Region II, use variable $X = (1 - r)/\epsilon$. Boundary condition $u'(1) = \tan\theta_0$ means $u = \epsilon U_0(X) + \ldots$, and

$$\left(\frac{U_0'}{[1 + (U_0')^2]^{1/2}}\right)' = U_0, \quad U_0'(0) = \tan\theta_0, \quad U_0'(\infty) = 0.$$

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Clever change of variables: use independent variable ψ defined as slope of profile, i.e. $\tan\psi = -U_0'$, so that

$$\frac{dX}{d\psi} = -\frac{\cos\psi}{U_0}, \quad \frac{dU_0}{d\psi} = -\frac{\sin\psi}{U_0}.$$

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Solutions to second is $U_0 = \sqrt{2(1-\cos t)}$, and first has solution

$$X = \frac{1}{2}\int_\psi^{\theta_0} \frac{\cos t}{\sin(t/2)}\,dt = \ln\tan\theta_0/4 + 2\cos\theta_0/2 - \ln\tan\psi/4 - 2\cos\psi/2.$$

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Need behavior as $X \to \infty$, same as $\psi \to 0$, gives

$$U_0(\psi) \sim \psi, \quad X(\psi) \sim \ln\tan\theta_0/4 + \cos\theta_0/2 - \ln\psi/4,$$

Thus as $X \to \infty$, have

$$U_0 \sim 4\tan(\theta_0/2)e^{-4\sin^2\theta_0/4}e^{-X} \equiv \gamma e^{-1/\epsilon}e^{r/\epsilon}.$$

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Finally, matching insists $C\nu_0(\epsilon)\epsilon^{1/2}e^{r/\epsilon}/\sqrt{2\pi} \sim \gamma e^{-1/\epsilon}e^{r/\epsilon}$, therefore $C\nu_0(\epsilon) = \gamma\sqrt{2\pi/\epsilon}e^{-1/\epsilon}$.

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In general, boundary layers on both boundaries, so use $X = (x \pm 1)/\epsilon$, and let $y = Y_0(X) + \epsilon Y_1(X) + \ldots$. Leading order solution is

$$Y_0 = \begin{cases} c + (a - c)e^{-X}, & \text{(left)}, \\ c + (b - c)e^{X}, & \text{(right)} \end{cases}$$

but nothing determines c, even at further orders Y_n!

Effect of transcendentally small terms,cont.

Resolution: expand outer $y = y_0 + \epsilon y_1 + \ldots + z_0(x;\epsilon) + \ldots$ where z_0 is transcendentally small, e.g. $z_0 \sim e^{-x/\epsilon}$. In this case, we have dominant balance $\epsilon z_0'' - x z_0' = 0$, whose solution is

$$z_0 = B(\epsilon) + A(\epsilon) \int_{-1}^{x} e^{s^2/2\epsilon} ds.$$

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To match, write in terms of inner variable and re-expand for small ϵ,

$$\int_{-1}^{-1+\epsilon X} e^{s^2/2\epsilon} ds \sim \epsilon \left(\frac{e^{(-1+\epsilon X)^2/2\epsilon}}{-1+\epsilon X} + e^{1/2\epsilon} \right) \sim \epsilon(1 - e^{-X}) e^{1/2\epsilon}.$$

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The exponentially decaying terms in the leading order boundary layers now have something to match onto; this gives $\epsilon A(\epsilon) e^{1/2\epsilon} = c - a$ for left boundary layer and $\epsilon A(\epsilon) e^{1/2\epsilon} = b - c$ for right, with $B(\epsilon) = -\epsilon A(\epsilon) e^{1/2\epsilon}$ for both. Elimination yields $c = (a+b)/2$.

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To illustrate this, consider the Lagerstrom model

$$y'' + \frac{2}{x}y' + \epsilon y y' = 0, \quad y(1) = 0, \quad y(\infty) = 1.$$

Expanding $y = y_0 + \epsilon y_1 + \ldots$, $y_0'' + 2y_0'/x = 0$, so that $y_0 = C_0(1 - 1/x)$. Continuing, $y_1'' + 2y_1/x + 1/x^2 - 1/x^3 = 0$ with $y_1(0) = 0$, so that

$$y_1 = c_1(1 - 1/x) + \ln x - \ln x/x.$$

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For large x, use variable $X = x\epsilon$, $y = Y(X)$ and expand in powers of ϵ. Then $Y_0 = 1$ and $Y_1'' + (2/X + 1)Y_1' = 0$ where $Y_1(\infty) = 0$. The solution is

$$Y_1 = A_1 E_2(X), \quad E_2 = \int_X^\infty e^{-t} t^{-2} dt.$$

Matching will need behavior for small X, found by integration by parts as

$$E_2 \sim 1/X + \ln X + (\gamma - 1) - X/2 + \mathcal{O}(X^2)$$

Switchback terms,cont.

Matching compares

$$y \sim (1 - \epsilon/X) + \epsilon c_1(1 - \epsilon/X) - \epsilon \ln X + \epsilon \ln \epsilon$$

to

$$Y \sim 1 + A_1(\epsilon/X + \epsilon \ln X + \epsilon(\gamma - 1)),$$

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Resolution: amend expansion $y = y_0 + \epsilon \ln \epsilon y^* + \epsilon y_1 + \ldots$, where y^* solves

$$(y^*)'' + (2/x)(y^*)' = 0, \quad y^*(1) = 0,$$

thus $y^* = c^*(1 - 1/x)$. Effect is new term when matching $c^* \epsilon \ln \epsilon (1 - \epsilon/x)$, which counterbalances existing $\epsilon \ln \epsilon$ term by the choice $c^* = -1$.

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Physical relevance: surface shear stress is

$$y'(1) \sim 1 - \epsilon \ln \epsilon - (\gamma + 1)\epsilon.$$

Triple deck

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For inner expansion(s), let $X = x/\epsilon^\alpha$, so $y = Y(X)$ solves

$$\epsilon^{3-2\alpha} Y'' + \epsilon^{2\alpha} X^3 Y + (\epsilon^{3\alpha} X^3 - \epsilon)Y = 0.$$

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With $\alpha = 1/2$, let $y = W(Z)$ with $x = Z/\epsilon^{1/2}$. Then $X^3 W_0' - W_0 = 0$, so that $W_0 = c_0 e^{-1/2X^2}$. Note that $W_0(0) = 0$ so this automatically matches the behavior of the first boundary layer for $X \to \infty$. Also,
$Y_0(\infty) = c_0 = y_0(0) = Be$.