Asymptotic Methods

Higher dimensional and moving internal layers

Moving internal layers

Many time-dependent PDEs have internal layers which move as a response to the outer solutions surrounding them. This case is handled using a moving, scaled coordinate system like $z = (x - x_0(t))/\epsilon^{\alpha}$, where $x_0(t)$ is the position of the layer at time t.

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$$u_t + J(u)_x = \epsilon u_{xx}, \quad -\infty < x < \infty,$$

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The outer solution here is prescribed as $u = u_-$ for $x < x_0(t)$ and $u = u_+$ for $x > x_0(t)$. Using $z = (x - x_0(t))/\epsilon$ as the new spatial variable gives

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Matching gives $\lim_{z\to\infty} u_0(z) = u_\pm$ so

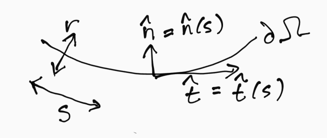
$$-x_0'(t)u_- + J(u_-) = C = -x_0'(t)u_+ + J(u_+)$$

or

$$x_0'(t) = \frac{J(u_+) - J(u_-)}{u_+ - u_-}, \quad \text{(Rankine-Hugoniot condition)}$$

Higher dimensional boundary layers

Problems in two spatial dimensions can have boundary layers which are curved; need a coordinate system (r, s) fitted to geometry



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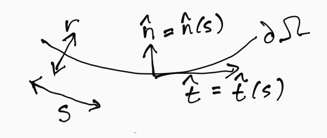

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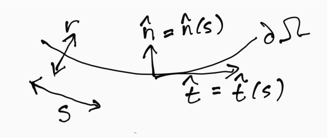

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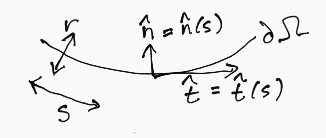

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$$\nabla s = \hat{\mathbf{t}}(1 - \kappa r), \quad \nabla r = \hat{\mathbf{n}}.$$

Laplacian of r, s obtained by taking divergence

$$\Delta r = d\hat{\mathbf{n}}/ds \cdot \nabla s = -\kappa/(1 - \kappa r), \quad \Delta s = \kappa'(s)r/(1 - \kappa r)^3.$$

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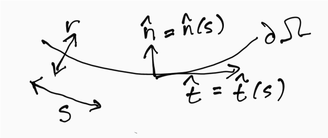

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Finally, for a function $u(x) = u(r, s)$,

$$\Delta u = u_{rr}|\nabla r|^2 + u_{ss}|\nabla s|^2 + 2u_{sr}\nabla s\cdot\nabla r + u_r\Delta r + u_s\Delta s = u_{rr} - \frac{\kappa u_r}{1 - \kappa r} + \frac{1}{1 - \kappa r}\left(\frac{u_s}{1 - \kappa r}\right)_s$$

Example: diffusion layer

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The outer solution away from the boundary is simply $V = 0$. The inner uses fitted coordinates and solves

$$\epsilon \left[V_{rr} - \frac{\kappa V_r}{1-\kappa r} + \frac{1}{1-\kappa r}\left(\frac{V_s}{1-\kappa r}\right)_s \right] - V = 0, \quad V(r=0,s) = u_b(s).$$

Diffusion layer,cont.

Introduce stretched coordinate $z = r/\epsilon^{1/2}$ so equation becomes $V_{zz} - \kappa\epsilon^{1/2}V_z - V = \mathcal{O}(\epsilon)$. Expand $V = V_0 + \epsilon^{1/2}V_1 + \ldots$ so

$$V_{0zz} - V_0 = 0, \quad V_0(0,s) = u_b, \quad V_0(\infty,s) = 0.$$

whose solution is $V_0 = u_b e^{-z}$.

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It follows that $U \sim (u_b/\sigma)[e^{-x/\epsilon^{1/2}} + (\kappa r/2)e^{-r/\epsilon^{1/2}}]$, whose inverse Laplace transform is

$$u(x,t) \sim u_b(1 + \kappa(s)r/2)\text{erfc}(r/(2\sqrt{Dt})).$$

A moving, multidimensional internal layer

The Allen-Cahn equation is

$$\epsilon^2 u_t = \epsilon^2 \Delta u + 2u(1 - u^2), \quad x \in \mathbb{R}^2.$$

where initial condition is $u = -1$ outside a closed curve $\Gamma(0)$ and $u = 1$ inside.
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In this problem, the outer solution is always $u = \pm 1$. The inner solution uses fitted, stretched coordinates (z, s, t) where $z = r/\epsilon$ and $r = r(x, t)$, $s = s(x, t)$; equation is

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$$u_{1zz} + u_{1z} - 3u_0^2 u_{1z} = u_{0z} r_t + \kappa u_{0z}.$$

It is not hard to show that the linear operator on the left is self-adjoint and has a nullspace spanned by eigenfunction $u_0'(z)$.

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The Fredholm alternative says that

$$\int_{-\infty}^{\infty} (u_{0z} r_t + \kappa u_{0z}) u_{0z} \, dz = 0$$

so that inward normal velocity is $r_t = -\kappa$.