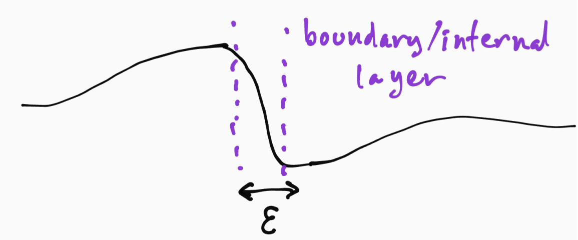
Asymptotic Methods

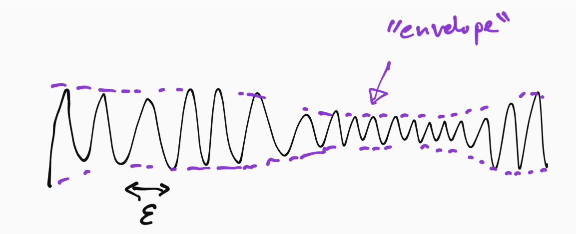
Multiple scale methods

The need for multiple scales

For boundary layers, a secondary scale (inner variable) was used in the method of matched asymptotic expansions.



For problems which involve rapid oscillation, multiple scale methods are used to determine the macroscopic behavior in terms of “envelope” or “amplitude” equations.



Example: damped oscillator

Consider

$$y'' + \epsilon y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

The exact solution is

$$\frac{1}{\sqrt{1-\epsilon^2/4}} e^{\epsilon t/2} \sin(t\sqrt{1-\epsilon^2/4}).$$

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What if a naive expansion is tried? Let $y = y_0 + \epsilon y_1 + \ldots$, giving

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so that $y_0 = \sin t$.

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so that $y_0 = \sin t$. At next order,

$$y_1'' + y_1 = -\cos t, \quad y_1(0) = 0, \quad y_1'(0) = 0,$$

so that $y_1 = -(t/2)\sin t$. Big problem: expansion is disordered when $t = \mathcal{O}(\epsilon^{-1})$.

Damped oscillator, cont.

Resolution: $y(t) = y_0(t, T) + \epsilon y_1(t, T) + \ldots$, where $T = \epsilon t$. Note

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T}, \quad \frac{d}{dt} \to \frac{\partial^2}{\partial^2 t} + 2\epsilon \frac{\partial^2}{\partial t \partial T} + \epsilon^2 \frac{\partial^2}{\partial^2 T},$$

so equation becomes

$$y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT} + \epsilon(y_t + \epsilon y_T) = 0.$$

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Leading order solves (easy) problem

$$y_{0tt} + y_0 = 0, \quad y_0(0,0) = 0, \quad y_{0t}(0,0) = 1,$$

so that $y_0 = A(T) \sin t + B_1(T) \cos t$ where $A(0) = 1$ and $B(0) = 0$.

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$$y_{1tt} + y_0 = -2y_{0tT} - y_{0t}, \quad y_1(0,0) = 0, \quad y_{1t}(0,0) = -y_{0T}(0,0).$$

whose solution is

$$y_1 = A_1(T) \sin t + B_1(t) \cos t - \frac{1}{2}(2B' + B)t \sin t - \frac{1}{2}(2A' + A)t \cos t.$$

Still growing ("secular") terms? Not if we choose $2A' + A = 0$ and $2B' + B = 0$, leading to $A = \exp(-T/2)$ and $B = 0$.

Rayleigh oscillator

Consider

$$y'' - \epsilon[1 - \frac{1}{3}(y')^2]y' + y = 0.$$

As before, let $y(t) = y_0(t, T) + \epsilon y_1(t, T) + \dots$. Leading order is similar, giving $y_0 = A(T)e^{it} + \text{c.c.}$, where c.c. represents the complex conjugate of the previous terms.

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Next order solves

$$(\partial_t^2 + 1)y_1 = 2y_{0tT} + [1 - \frac{1}{3}(y_{0t})^2]y_{0t} = 2uA'e^{it} + iAe^{it} - \frac{1}{3}A^3e^{3it} + i|A|^2Ae^{it} + \text{c.c.}$$

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Secular terms are those which are resonant, i.e. proportional to e^{it}. Elimination gives

$$A' = \frac{1}{2}[A - |A|^2A].$$

Letting $A = R(T)e^{i\theta(T)}$, above is same as

$$R' = \frac{1}{2}[R - R^3], \quad \theta' = 0.$$

For nonzero initial conditions $R \to 1$ as $T \to \infty$, and solution approaches limit cycle $y \sim e^{i\theta(0)+t} + \text{c.c.}$.

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$$-A_0 \exp[i(\sqrt{a} + 1)t] - A_0 \exp[i(\sqrt{a} - 1)t] + c.c$$

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For y_2, secular terms appear when $\sqrt{a} \pm 2 = \pm\sqrt{a}$, so $a = 1$. In general, secular terms appear in n-th equation in hierarchy if $\sqrt{a} \pm n = \pm\sqrt{a}$, or $a = n^2/4$.

Parametric resonance and the Matheiu equation,cont.

Want to explore dynamics near first resonance value, so let $a = 1/4 + a_1\epsilon$; with multiple scale ansatz $y(t) = y_0(t, T) + \epsilon y_1(t, T) + \ldots$, leading order is $y_{0tt} + \frac{1}{4}y_0 = 0$, so $y_0 = A(T)e^{it/2} + \text{c.c.}$

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Next order is

$$y_{1tt} + \frac{1}{4}y_1 = -[a_1 A i A' + \overline{A}]e^{it/2} - Ae^{3it/2} + \text{c.c.}.$$

Eliminating secular terms leads to $iA' = a_1 A - \overline{A}$; decomposing $A = B + iC$

$$\begin{pmatrix} B \\ C \end{pmatrix}' = \begin{pmatrix} 0 & -a_1 + 1 \\ a_1 + 1 & 0 \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}$$

Parametric resonance and the Matheiu equation,cont.

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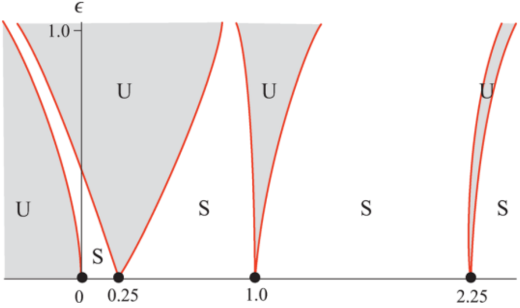
The eigenvalues are $\lambda = \pm\sqrt{1 - a_1^2}$, so unstable if $|a_1| < 1$.



A more general approach: the Poincare-Linstedt method

Consider Duffing oscillator

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We could try $y = y(t, \epsilon t)$ as before, leading to $y \sim \frac{1}{2} e^{i[1+3\epsilon/8]t}$. This solution does avoid secular growth, but eventually becomes out of phase, since the true frequency is not exactly $1 + 3\epsilon/8$.

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Alternately, we could allow the frequency to have initially arbitrary ϵ-dependence:

$$y = y(T; \epsilon), \quad T = \omega(\epsilon), \quad y \text{ periodic in } T.$$

In terms of T variable, problem is $\omega^2(\epsilon)y'' + y + \epsilon y^3 = 0$. Expand both $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots$ and $\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$.

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Leading order problem is

$$\omega_0 y_0'' + y_0 = 0, \quad y_0(0) = 1, \quad y_0'(0) = 0, \quad y_0(T) = y_0(T + 2\pi),$$

whose solution is $y_0 = \cos T$ and $w_0 = 1$.

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Next order problem is

$$y_1'' + y_1 = (2\omega_1 - 3/4)\cos T - \frac{1}{4}\cos(3T), \quad y_1(0) = 0, \quad y_1'(0) = 0, \quad y_1(T) = y_1(T + 2\pi$$

Periodicity means $\cos T$ secular term must vanish, so $\omega_1 = 3/8$, and then

$$y_1 = \frac{1}{32}(\cos 3T - \cos T).$$