Asymptotic Methods

Multiple scale methods
The need for multiple scales

For boundary layers, a secondary scale (inner variable) was used in the method of matched asymptotic expansions.

For problems which involve rapid oscillation, multiple scale methods are used to determine the macroscopic behavior in terms of “envelope” or “amplitude” equations.
Consider

\[ y'' + \epsilon y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1. \]

The exact solution is

\[ \frac{1}{\sqrt{1 - \epsilon^2/4}} e^{\epsilon t/2} \sin(t \sqrt{1 - \epsilon^2/4}). \]

Notice that oscillations occur on a scale \( t \sim \mathcal{O}(1) \), whereas the amplitude decays on a slower timescale \( T = \epsilon t \).
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so that \( y_0 = \sin t \). At next order,

\[ y_1'' + y_1 = -\cos t, \quad y_1(0) = 0, \quad y_1'(0) = 0, \]

so that \( y_1 = -(t/2) \sin t \). Big problem: expansion is disordered when \( t = O(\epsilon^{-1}) \).
Resolution: \( y(t) = y_0(t, T) + \epsilon y_1(t, T) + \ldots \), where \( T = \epsilon t \). Note

\[
\frac{d}{dt} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T}, \quad \frac{d}{dt} \to \frac{\partial^2}{\partial^2 t} + 2\epsilon \frac{\partial^2}{\partial t \partial T} + \epsilon^2 \frac{\partial^2}{\partial^2 T},
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so equation becomes

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y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT} + \epsilon(y_t + \epsilon y_T) = 0.
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Leading order solves (easy) problem

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y_{0tt} + y_0 = 0, \quad y_0(0, 0) = 0, \quad y_0t(0, 0) = 1,
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so that \( y_0 = A(T) \sin t + B_1(T) \cos t \) where \( A(0) = 1 \) and \( B(0) = 0 \).
Damped oscillator, cont.

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\[
y_{1tt} + y_0 = -2y_{0t} T - y_{0t}, \quad y_1(0, 0) = 0, \quad y_{1t}(0, 0) = -y_{0T}(0, 0).
\]

whose solution is

\[
y_1 = A_1(T) \sin t + B_1(t) \cos t - \frac{1}{2} (2B' + B) t \sin t - \frac{1}{2} (2A' + A) t \cos t.
\]

Still growing ("secular") terms? Not if we choose \( 2A' + A = 0 \) and \( 2B' + B = 0 \), leading to \( A = \exp(-T/2) \) and \( B = 0 \).
Consider
\[ y'' - \epsilon[1 - \frac{1}{3}(y')^2]y' + y = 0. \]

As before, let \( y(t) = y_0(t, T) + \epsilon y_1(t, T) + \ldots \). Leading order is similar, giving \( y_0 = A(T)e^{it} + \text{c.c.} \), where c.c. represents the complex conjugate of the previous terms.
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Next order solves

\[ (\partial_t^2 + 1)y_1 = 2y_0T + \left[1 - \frac{1}{3}(y_0)^2\right]y_0 = 2uA'e^{it} + iAe^{it} - \frac{1}{3}A^3e^{3it} + i|A|^2Ae^{it} + \text{c.c.} \]

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\[ A' = \frac{1}{2} [A - |A|^2 A]. \]

Letting \( A = R(T) e^{i\theta(T)} \), above is same as

\[ R' = \frac{1}{2} [R - R^3], \quad \theta' = 0. \]

For nonzero initial conditions \( R \to 1 \) as \( T \to \infty \), and solution approaches limit cycle \( y \sim e^{i\theta(0)t} + \text{c.c.} \).
Mathieu equation is

$$y_{tt} + (a + 2\epsilon \cos t)y = 0.$$
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First try regular perturbation series \( y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots \), produces hierarchy of problems

\[ y_{0tt} + ay_0 = 0 \]
\[ y_{1tt} + ay_1 = -2y_0 \cos t \]
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It follows \( y_0 = A_0 \exp(i\sqrt{a}t) + \text{c.c.} \), and the right hand side of \( y_1 \) equation is
\[ -A_0 \exp[i(\sqrt{a} + 1)t] - A_0 \exp[i(\sqrt{a} - 1)t] + \text{c.c} \]
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For \( y_2 \), secular terms appear when \( \sqrt{a} \pm 2 = \pm\sqrt{a} \), so \( a = 1 \). In general, secular terms appear in \( n \)-th equation in hierarchy if \( \sqrt{a} \pm n = \pm\sqrt{a} \), or \( a = n^2/4 \).
Want to explore dynamics near first resonance value, so let $a = 1/4 + a_1 \epsilon$; with multiple scale ansatz $y(t) = y_0(t, T) + \epsilon y_1(t, T) + \ldots$, leading order is $y_{0tt} + \frac{1}{4} y_0 = 0$, so $y_0 = A(T)e^{it/2} + c.c.$
Parametric resonance and the Mathieu equation, cont.

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Next order is

$$y_{1tt} + \frac{1}{4} y_1 = -[a_1 A i A' + A]e^{it/2} - Ae^{3it/2} + \text{c.c.}.$$  

Eliminating secular terms leads to $iA' = a_1 A - A$; decomposing $A = B + iC$

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The eigenvalues are $\lambda = \pm \sqrt{1 - a_1^2}$, so unstable if $|a_1| < 1$. 

![Graph showing phase portraits](image)
Consider Duffing oscillator

\[ y'' + y + \epsilon y^3 = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

We could try \( y = y(t, \epsilon t) \) as before, leading to \( y \sim \frac{1}{2} e^{i[1 + 3\epsilon/8]t} \). This solution does avoid secular growth, but eventually becomes out of phase, since the true frequency is not exactly \( 1 + 3\epsilon/8 \).
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Alternately, we could allow the frequency to have initially arbitrary \( \epsilon \)-dependence:

\[ y = y(T; \epsilon), \quad T = \omega(\epsilon), \quad y \text{ periodic in } T. \]

In terms of \( T \) variable, problem is \( \omega^2(\epsilon)y'' + y + \epsilon y^3 = 0 \). Expand both \( y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots \) and \( \omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots \).
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Leading order problem is
\[ \omega_0 y_0'' + y_0 = 0, \quad y_0(0) = 1, \quad y_0'(0) = 0, \quad y_0(T) = y_0(T + 2\pi), \]
whose solution is \( y_0 = \cos T \) and \( \omega_0 = 1 \).
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Next order problem is

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y''_1 + y_1 = (2\omega_1 - 3/4) \cos T - \frac{1}{4} \cos(3T), \quad y_1(0) = 0, \quad y'_1(0) = 0, \quad y_1(T) = y_1(T + 2\pi),
\]

Periodicity means \( \cos T \) secular term must vanish, so \( \omega_1 = 3/8 \), and then \( y_1 = \frac{1}{32} (\cos 3T - \cos T) \).