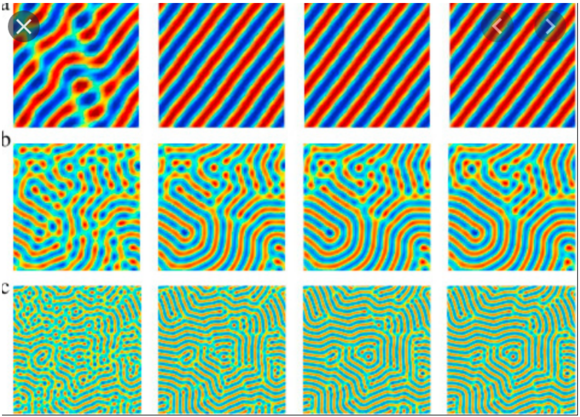
Asymptotic Methods

Multiple scale techniques for PDEs: waves and patterns

The Swift-Hohenberg equation

Seminal prototype for pattern formation (convection rolls, optical instabilities, biological patterns,...)

$$u_t = -(\Delta + 1)^2 u + N(u).$$



Localized states of SH

Consider one dimensional steady state problem

$$0 = -(1+\partial_x)^2 u - \epsilon u + u^3, \quad u(\pm\infty) = 0.$$

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Simplest multiple scale solution: introduce slow scale $y = \epsilon^\alpha x$, so that $u(x, y)$ solves

$$u_{xxxx} + 4\epsilon^\alpha u_{xxxy} + 6\epsilon^{2\alpha} u_{xxyy} + 4\epsilon^{3\alpha} u_{xyyy} + \epsilon^{4\alpha} u_{yyyy} + 2u_{xx} + 4\epsilon^\alpha u_{xy} + \epsilon^{2\alpha} u_{yy} = -\epsilon u + u^3.$$

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$$u = \epsilon^{1/2} u_1 + \epsilon u_2 + \epsilon^{3/2} u_3 + \dots$$

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Leading order problem is $\mathcal{L}u \equiv (1+\partial_x^2)^2 u_1 = 0$, so $u_1 = A(y)\cos(x) + B(y)\sin(x)$. For simplicity, set $B = 0$. Need $A(\pm\infty) = 0$ to satisfy side conditions.

Localized states of SH, cont.

Order ϵ terms produce nothing consequential, and $\mathcal{O}(\epsilon^{3/2})$ terms gives

$$\mathcal{L}u_3 = \begin{cases} (-2A'' - A + (3/4)A^3)\cos x, & \alpha \neq 1/2 \\ (4A'' - A + (3/4)A^3)\cos x, & \alpha = 1/2 \end{cases} + \text{non-secular terms.}$$

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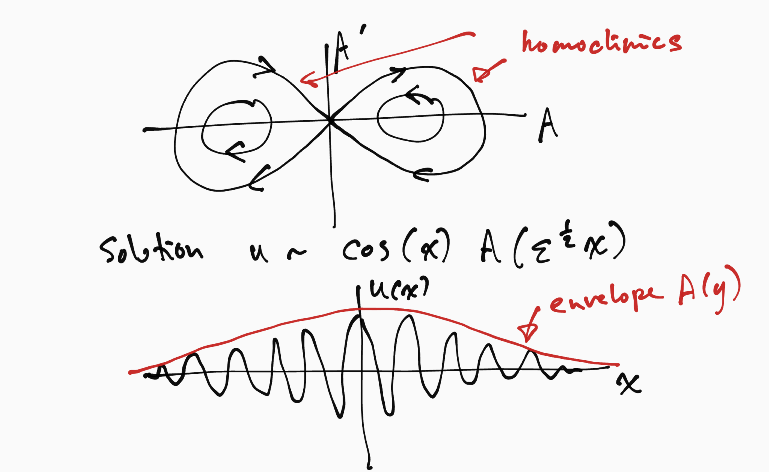
Only the case $\alpha = 1/2$ yields a system $4A'' - A + (3/4)A^3 = 0$ with a homoclinic orbit



Pattern formation and the Ginzburg-Landau equation

Consider an abstract evolution equation for $u(x,t)$ of the form

$$u_t = \mathcal{L}_\mu u + N(u), \quad N(u) = \mathcal{O}(u^2).$$

Suppose the linearized equation $u_t = \mathcal{L}_\mu u$ admits solutions $\epsilon(\sigma t + ikx)$ with $\sigma = \sigma(k,\mu)$ so that
(1) Re $\sigma < 0$ for all k if $\mu < 0$, and
(2) exists wavenumber $k_c > 0$ so Re $\sigma(k_c,\mu) > 0$ if $\mu > 0$.

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For example, $\mathcal{L} = \mu - (1 + \partial^2/\partial x^2)^2$ has $\sigma = \mu - (1-k^2)^2$. Set $\mu = r\epsilon^2$, $k_c = 1$, so that unstable modes have form $k = 1 + \epsilon K$, where

$$K \in (-\frac{1}{2}\sqrt{r}, \frac{1}{2}\sqrt{r}) + \mathcal{O}(\epsilon),$$

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How do unstable modes evolve?

$$u \sim \exp[i(1+\epsilon K)x + \epsilon^2(r-4K)^2 t] = \exp(ix)\exp[iK(\epsilon x) + (r-4K^2)(\epsilon^2 t)] = A(\epsilon x, \epsilon^2 t)e^{ix}.$$

This suggests using a multiple scale ansatz $u \sim \epsilon^\alpha U(x, X, T)$ where $X = \epsilon x$ and $T = \epsilon^2 t$, with α determined by nonlinearity.

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Goal: find amplitude evolution equation, known as the Ginzburg-Landau equation.

GL equation for Swift-Hohenberg

Consider dynamic SH equation

$$u_t = \mu u - (1 + \partial^2/\partial x^2)^2 - u^3, \quad \mu = r\epsilon^2.$$

(note difference with steady-state problem: sign of r is positive here)
With multiple scale ansatz $u = \epsilon U(x, X, T)$, problem is

$$\epsilon^2 U_T = -U_{xxxx} - 2U_{xx} - 4\epsilon U_{xX} - 2\epsilon^2 U_{XX} - U_{xxxx} - 4\epsilon U_{xxxX} - 6\epsilon^2 U_{xxXX} + r\epsilon^2 U - \epsilon^2 U^3 + \mathcal{O}(\epsilon^3)$$

Expanding in powers of ϵ, leading order problem is

$$-U_{0xxxx} - 2U_{0xx} - U_0 \equiv \mathcal{L}_0 U_0 = 0,$$

whose solution is $U_0 = A(X, T)e^{ix} + \text{c.c.}$.

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whose solution is $U_0 = A(X, T)e^{ix} + \text{c.c.}$.

As in steady state analysis, nothing of consequence occurs at $\mathcal{O}(\epsilon^2)$, and at order ϵ^3, get

$$\mathcal{L}_0 U_2 = (A_T - rA - 4A_{XX} + 3|A|^2 A)e^{ix} + \text{c.c.} + \text{non-secular}.$$

Result is Ginzburg-Landau equation

$$A_T = 4A_{XX} + rA - 3|A|^2 A,$$

describing propagation of pattern into unstable state.

Amplitude equation for nonlinear waves

Consider “Sine -Gordon” equation

$$\psi_{tt} = c^2\psi_{xx} - \omega_0^2 \sin\psi, \quad -\infty < x < \infty$$

The linearized equation $\psi_{tt} = c^2\psi_{xx} - \omega_0^2\psi$ has a dispersion relation $\omega^2 = \omega_0^2 + c^2k^2$.

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Here, we impose a slow modulation on an underlying wave of the form $e^{ikx-i\omega t}$, which has scales $X = \epsilon x$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$.

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Expanding $\psi \sim \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \ldots$, leading order is

$$\psi_{1tt} - c^2\psi_{1xx} + \omega_0^2\psi_1 \equiv \mathcal{L}\psi = 0.$$

Impose specific wavelength k, so $\psi_1 = A(X, T_1, T_2)e^{ikx-i\omega t}$.

Nonlinear waves cont.

For $\mathcal{O}(\epsilon^2)$, have

$$\mathcal{L}\psi_2 = (2i\omega A_{T_1} + 2ic^2 kA_X)e^{ikx-i\omega t} + \text{c.c.},$$

so to avoid secular terms,

$$A_{T_1} + v_g A_X = 0, \quad v_g = c^2 k/\omega = \omega'(k).$$

In other words, amplitude propagates at group velocity. Solution to this transport equation is $A = (X - v_g T_1, T_2)$.

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For $\mathcal{O}(\epsilon^3)$, have

$$\mathcal{L}\psi_3 = (-A_{T_1 T_1} 2i\omega A_{T_2} + c^2 A_{XX} + \frac{1}{2}\omega_0^2 |A|^2 A)e^{ikx - i\omega t} + \text{c.c.} + \text{non-secular terms}$$

so to avoid secular terms,

$$A_{T_2} = -\frac{1}{2i\omega}\left((c^2 v_g^2)A_{XX} + \frac{1}{2}\omega_0^2 |A|^2 A\right) = \frac{i}{2}\omega''(k)A_{XX} + \frac{i}{4}\frac{\omega_0^2}{\omega}|A|^2 A.$$

This is the nonlinear Schrödinger equation; the term $\omega''(k)$ is the group velocity dispersion.