Asymptotic Methods
Preliminaries: motivation and order notation

Introduction

How can we understand a difficult mathematical model?

- Exact solutions (rare)

Introduction

How can we understand a difficult mathematical model?

- Exact solutions (rare)
- Rigorous theorems (often not constructive)

Introduction

How can we understand a difficult mathematical model?

- Exact solutions (rare)
- Rigorous theorems (often not constructive)
- Numerical simulation (limited to specific inputs)

Is there a middle ground?

Introduction

How can we understand a difficult mathematical model?

- Exact solutions (rare)
- Rigorous theorems (often not constructive)
- Numerical simulation (limited to specific inputs)

Is there a middle ground?

Limiting cases are often perfectly tractable!

This happens when a parameter is close to a limiting value, for example small Reynolds number in fluid flow, velocity near speed of light, large numbers in probability

Preliminary example

Consider launching a rocket, whose trajectory is subject to both gravity and friction

$$\ddot{x} = -\frac{GMm}{(x+R)^2} - k\dot{x}.$$

This is second order, not integrable, and nonlinear - no exact solution is possible.

Preliminary example

Consider launching a rocket, whose trajectory is subject to both gravity and friction

$$\ddot{x} = -\frac{GMm}{(x+R)^2} - k\dot{x}.$$

This is second order, not integrable, and nonlinear - no exact solution is possible.

Nondimensionalize $x' = x/R$, $t' = t/(R^3/GM)^{1/2}$, giving

$$\ddot{x} = -\frac{1}{(x+1)^2} - \epsilon\dot{x}, \quad \epsilon = k\sqrt{\frac{R^3}{GMm}} \ll 1.$$

Preliminary example

Consider launching a rocket, whose trajectory is subject to both gravity and friction

$$\ddot{x} = -\frac{GMm}{(x+R)^2} - k\dot{x}.$$

This is second order, not integrable, and nonlinear - no exact solution is possible.

Nondimensionalize $x' = x/R$, $t' = t/(R^3/GM)^{1/2}$, giving

$$\ddot{x} = -\frac{1}{(x+1)^2} - \epsilon\dot{x}, \quad \epsilon = k\sqrt{\frac{R^3}{GMm}} \ll 1.$$

Consider $x(0) = 0$ and $x'(0) = \sqrt{2}$. Setting $\epsilon = 0$, the equation is integrable:

$$\frac{\dot{x}^2}{2} \approx \frac{1}{x+1}, \quad x(t) \approx x_0(t) = \eta^{2/3} - 1, \quad \eta \equiv 3\sqrt{2}t/2 + 1.$$

But what about effect of friction?

Preliminary example

Consider launching a rocket, whose trajectory is subject to both gravity and friction

$$\ddot{x} = -\frac{GMm}{(x+R)^2} - k\dot{x}.$$

This is second order, not integrable, and nonlinear - no exact solution is possible.

Nondimensionalize $x' = x/R$, $t' = t/(R^3/GM)^{1/2}$, giving

$$\ddot{x} = -\frac{1}{(x+1)^2} - \epsilon\dot{x}, \quad \epsilon = k\sqrt{\frac{R^3}{GMm}} \ll 1.$$

Consider $x(0) = 0$ and $x'(0) = \sqrt{2}$. Setting $\epsilon = 0$, the equation is integrable:

$$\frac{\dot{x}^2}{2} \approx \frac{1}{x+1}, \quad x(t) \approx x_0(t) = \eta^{2/3} - 1, \quad \eta \equiv 3\sqrt{2}t/2 + 1.$$

But what about effect of friction?

Example cont.

Incorporate correction term $x(t) = x_0(t) + \epsilon x_1(t)$, so that

$$\ddot{x}_0 + \epsilon \ddot{x}_1 = -\frac{1}{(x_0 + \epsilon x_1 + 1)^2} - \epsilon(\dot{x}_0 + \epsilon \dot{x}_1)$$

After Taylor expansion of the nonlinear term, retain only terms with ϵ:

$$\ddot{x}_1 = \frac{2x_1}{(x_0 + 1)^3} - \dot{x}_0 = \frac{2x_1}{\eta^3} - \sqrt{2}\eta^{-1/3}.$$

Example cont.

Incorporate correction term $x(t) = x_0(t) + \epsilon x_1(t)$, so that

$$\ddot{x}_0 + \epsilon \ddot{x}_1 = -\frac{1}{(x_0 + \epsilon x_1 + 1)^2} - \epsilon(\dot{x}_0 + \epsilon \dot{x}_1)$$

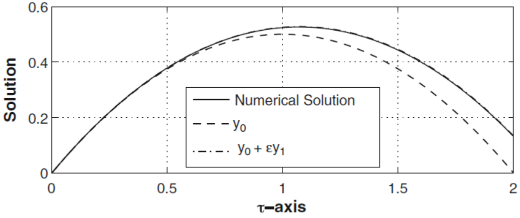
After Taylor expansion of the nonlinear term, retain only terms with ϵ:

$$\ddot{x}_1 = \frac{2x_1}{(x_0 + 1)^3} - \dot{x}_0 = \frac{2x_1}{\eta^3} - \sqrt{2}\eta^{-1/3}.$$

This is linear and inhomogeneous, so we can solve explicitly

$$x_1 = c_1 \eta^{4/3} + c_2 \eta^{-1/3} - \frac{\sqrt{2}}{3}\eta^{5/3}.$$

Also, $x_1(0) = 0 = \dot{x}_1(0)$, leading to $c_1 = 16\sqrt{2}/39$ and $c_2 = -\sqrt{2}/13$.



$\epsilon = .1$

Order (Landau) notation

How to compare sizes of (dimensionless) quantities as a limit is approached?

Order (Landau) notation

How to compare sizes of (dimensionless) quantities as a limit is approached?

Definition

We say $f(x) = \mathcal{O}(g(x))$ as $x \to 0$ if there exists K, δ so that

$$|f(x)| \leq K|g(x)|$$

whenever $|x| < \delta$.

Definition

We say $f(x) = o(g(x))$ as $x \to 0$ if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0.$$

Definition

We say $f(x) \sim g(x)$ as $x \to 0$ if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1.$$

Order (Landau) notation, cont.

An easy characterization of big-O notation is the following:

Lemma

If $\lim_{x \to 0} \left| \frac{f(x)}{g(x)} \right| = m < \infty$, then $f(x) = \mathcal{O}(g(x))$.

Proof: existence of the limit means that for any small ε there exists some δ so that $|x| < \delta$ implies

$$\left| \left| \frac{f(x)}{g(x)} \right| - m \right| \leq \varepsilon,$$

which is the same as $|f(x)| \leq (|m| + \varepsilon)|g(x)|$.

Exercise: show the converse is not true!

Order (Landau) notation, cont.

An easy characterization of big-O notation is the following:

Lemma

If $\lim_{x \to 0} \left| \frac{f(x)}{g(x)} \right| = m < \infty$, then $f(x) = \mathcal{O}(g(x))$.

Proof: existence of the limit means that for any small ε there exists some δ so that $|x| < \delta$ implies

$$\left| \left| \frac{f(x)}{g(x)} \right| - m \right| \leq \varepsilon,$$

which is the same as $|f(x)| \leq (|m| + \varepsilon)|g(x)|$.

Exercise: show the converse is not true!

- It is useful to think of $\mathcal{O}$, o and $\sim$ like $\leq, <, =$.
- Similar definitions exist for $x \to x_0$ and $x \to \infty$.

Simple examples of order notation

Example 1: $\sin(x) = \mathcal{O}(x)$ as $x \to 0$ since $\lim_{x\to 0} \sin x/x = 1$.
This also means that $\sin(x) \sim x$ as $x \to 0$.

Simple examples of order notation

Example 1: $\sin(x) = \mathcal{O}(x)$ as $x \to 0$ since $\lim_{x\to 0} \sin x/x = 1$.
This also means that $\sin(x) \sim x$ as $x \to 0$.

Example 2: If $f(x)$ is real analytic and $P_n(x)$ is the n-th order Taylor polynomial about x_0, then $f(x) \sim P_n(x)$ as $x \to x_0$. This is true since remainder theorem implies

$$f(x) - P_n(x) = \mathcal{O}(|x - x_0|^{n+1}),$$

so that

$$\frac{f(x)}{P_n(x)} = 1 + \frac{\mathcal{O}(|x - x_0|^{n+1})}{P_n(x)}, \quad \text{which} \to 1 \text{ as } x \to x_0.$$

Simple examples of order notation

Example 1: $\sin(x) = \mathcal{O}(x)$ as $x \to 0$ since $\lim_{x\to 0} \sin x/x = 1$.
This also means that $\sin(x) \sim x$ as $x \to 0$.

Example 2: If $f(x)$ is real analytic and $P_n(x)$ is the n-th order Taylor polynomial about x_0, then $f(x) \sim P_n(x)$ as $x \to x_0$. This is true since remainder theorem implies

$$f(x) - P_n(x) = \mathcal{O}(|x - x_0|^{n+1}),$$

so that

$$\frac{f(x)}{P_n(x)} = 1 + \frac{\mathcal{O}(|x - x_0|^{n+1})}{P_n(x)}, \quad \text{which} \to 1 \text{ as } x \to x_0.$$

Example 3: For any positive integer n, $\exp(-1/x) = o(x^n)$ as $x \to 0^+$.
(note one sided limit)

Simple examples of order notation

Example 1: $\sin(x) = \mathcal{O}(x)$ as $x \to 0$ since $\lim_{x\to 0} \sin x/x = 1$.
This also means that $\sin(x) \sim x$ as $x \to 0$.

Example 2: If $f(x)$ is real analytic and $P_n(x)$ is the n-th order Taylor polynomial about x_0, then $f(x) \sim P_n(x)$ as $x \to x_0$. This is true since remainder theorem implies

$$f(x) - P_n(x) = \mathcal{O}(|x - x_0|^{n+1}),$$

so that

$$\frac{f(x)}{P_n(x)} = 1 + \frac{\mathcal{O}(|x - x_0|^{n+1})}{P_n(x)}, \quad \text{which} \to 1 \text{ as } x \to x_0.$$

Example 3: For any positive integer n, $\exp(-1/x) = o(x^n)$ as $x \to 0^+$.
(note one sided limit)

Example 4: For any positive integer n, $\ln x = o(x^n)$ as $x \to \infty$.

Simple examples of order notation

Example 1: $\sin(x) = \mathcal{O}(x)$ as $x \to 0$ since $\lim_{x \to 0} \sin x / x = 1$.
This also means that $\sin(x) \sim x$ as $x \to 0$.

Example 2: If $f(x)$ is real analytic and $P_n(x)$ is the n-th order Taylor polynomial about x_0, then $f(x) \sim P_n(x)$ as $x \to x_0$. This is true since remainder theorem implies

$$f(x) - P_n(x) = \mathcal{O}(|x - x_0|^{n+1}),$$

so that

$$\frac{f(x)}{P_n(x)} = 1 + \frac{\mathcal{O}(|x - x_0|^{n+1})}{P_n(x)}, \quad \text{which} \to 1 \text{ as } x \to x_0.$$

Example 3: For any positive integer n, $\exp(-1/x) = o(x^n)$ as $x \to 0^+$.
(note one sided limit)

Example 4: For any positive integer n, $\ln x = o(x^n)$ as $x \to \infty$.

A couple useful facts:

- $f(x) = \mathcal{O}(1)$ as $x \to x_0$ means $f(x)$ is locally bounded near x_0.
- $f(x) = o(1)$ as $x \to x_0$ means $f(x) \to 0$ as x_0 is approached.