Asymptotic Methods

Solvability conditions and eigenvalue problems

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Suppose that $L : V \to V$ is a linear mapping from a Hilbert space V to itself (e.g. a square matrix). If the adjoint of L has a nullspace spanned by $v_1, v_2, \ldots, v_k$, then necessary and sufficient conditions to solve $Lv = b$ are

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Remarks:

- Proof is easy: simply take the inner product of both sides with each v_i, and use the definition of adjoint.
- Notice that the solution of $Lx = b$ will not be unique, but rather the sum of a particular solution plus the nullspace of L.
- For boundary value problems, the boundary conditions also come into play.

Perturbation of finite dimensional linear problems

If A, B are nonsingular, symmetric, square matrices, then $(A + \epsilon B)x = y$ admits a regular expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$. Inserting this and equating powers of ϵ gives a hierarchy of linear problems

$$Ax_n = -Bx_{n-1}, \quad n = 1, 2, 3, \dots$$

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The first equation has solution $x_{-1} = cv$ where c is yet undetermined. The second equation can only be solved if the right had side is orthogonal to v, which means

$$v \cdot (y - cBv) = 0, \quad \text{or} \quad c = (y \cdot v)/(Bv \cdot v).$$

Notice this "selects" the value of c. This process can be repeated for other terms in the expansion.

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Expand both $x = x_0 + \epsilon x_1 + \dots$ and $\lambda = \lambda_0 + \epsilon \lambda_1 + \dots$, so that the leading order problem $Ax_0 = \lambda x_0$ is automatically solved. The next order gives

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The matrix $A - \lambda_0 I$ is necessarily singular, and therefore so is the adjoint $A^* - \lambda_0 I$. Letting x^* be the adjoint eigenvector solving $(A^* - \lambda_0 I)x^* = 0$, Fredholm implies

$$(\lambda_1 x_0 - Bx_0) \cdot x^* = 0, \quad \text{or} \quad \lambda_1 = \frac{x^* \cdot Bx_0}{x^* \cdot x_0}.$$

Solvability of boundary value problems

Consider the standard Sturm-Liouville boundary value problem for $u = u(x)$,

$$\mathcal{L}u \equiv (p(x)u')' + q(x)u = r(x), \quad u(a) = u_a, \quad u(b) = u_b.$$

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If the operator $\mathcal{L}$ is not singular (no zero eigenvalues), this problem has a unique solution.

On the other hand, if $\mathcal{L}$ has a nullspace spanned by the eigenfunction $v(x)$, then a solvability condition is derived by multiplying the equation by $v(x)$ and integrating over the domain. The result is

$$\int_a^b v(x)r(x)\,dx + \left[p(x)(u'(x)v(x) - u(x)v'(x))\right]_a^b = 0.$$

This is both a necessary and sufficient condition to find a solution.

Example: a perturbed SL eigenvalue problem

Consider

$$u'' + q(x)u = 0, \quad u(0) = 0 = u(L).$$

where $q(x) = \lambda + \epsilon a(x)$. For $\epsilon = 0$, this is a standard eigenvalue problem, which has solutions

$$u_0 = \sin(n\pi x/L), \quad \lambda_0 = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

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We can perturb around a particular eigenpair by expanding $\lambda = \lambda_0 + \epsilon\lambda_1 + \dots$ and $u = u_0 + \epsilon u_1 + \dots$, giving

$$\mathcal{L}u_1 \equiv u_1'' + \lambda_0 u_1 = -\lambda_1 u_0 - a(x)u_0.$$

Since λ_0 is an eigenvalue, $\mathcal{L}$ is singular (and self adjoint) with nullspace spanned by u_0. Solvability gives

$$\int_a^b (-\lambda_1 u_0 - a(x)u_0)u_0 \, dx = 0, \quad \text{or} \quad \lambda_1 = -\frac{\int_a^b a(x)u_0(x)^2 dx}{\int_a^b u_0(x)^2 dx}.$$

Example: diffusion through a slender body

Consider a domain which is a thin tube, described in cylindrical coordinates as $\{(r, Z)|0 < Z < L, r < R_0 f(z/L)\}$ where $R_0 \ll L$. Want to solve the steady state diffusion equation

$$\Delta T(r, Z) = T_{rr} + \frac{1}{r} T_r + T_{ZZ} = 0,$$
$$T(r, 0) = 0, \quad T(r, 1) = T_0, \quad \nabla T \cdot n = 0 \quad \text{on } r = f(z/L).$$

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Nondimensionalize $u = T/T_0$, $\rho = r/R_0$, $z = Z/L$, giving

$$u_{\rho\rho} + \frac{1}{\rho} u_\rho + \epsilon^2 u_{zz} = 0, \quad u(\rho, 0) = 0, \quad u(\rho, 1) = 1, \quad u_\rho = \epsilon^2 f'(z) u_z \quad \text{on } \rho = f(z).$$

where $\epsilon = R_0/L$.

Example: diffusion through a slender body, cont.

Expanding $u = u_0 + \epsilon u_1 + \dots$ leads to

$$\mathcal{L}u_0 = u_{0\rho\rho} + \frac{1}{\rho} = 0, \quad 0 \leq \rho < f(z), \quad u_{0\rho}(f(z), z) = 0,$$

$$\mathcal{L}u_1 = -u_{0zz} = 0 \leq \rho < f(z), \quad u_{1\rho}(f(z), z) = f'(z)u_{0z}.$$

The solution for the first is $u_0 = A(z)$. The linear operator $\mathcal{L}$ is self adjoint with respect to the inner product $\int_0^{f(z)} v_1(\rho)v_2(\rho)\rho d\rho$. The second equation therefore has a solvability condition obtained by multiplying by the eigenfunction 1 and integrating:

$$-\int_0^{f(z)} u_{0zz}\rho d\rho = \int_0^{f(z)} (\rho u_{1\rho})_\rho d\rho = \rho u_{1\rho}\Big|_0^{f(z)} = f(z)f'(z)u_{0z}.$$

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Thus $f(z)^2 A''(z) + f(z)f'(z)A'(z) = 0$, which can be compactly written as

$$\left[f(z)^2 A_z\right]_z = 0.$$

This is just one dimensional steady state diffusion with variable diffusivity. Using $A(0) = 0$ and $A(1) = 1$, one can solve by direct integration

$$A(z) = \frac{\int_0^z f(z')^2 dz'}{\int_0^1 f(z')^2 dz'}.$$