Asymptotic Methods

Method of stationary phase

Highly oscillatory integrals

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$$\frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}\bigg|_{t=a}^{t=b} - \frac{1}{ix\psi'(t)}\int_a^b \left(\frac{f(t)}{\psi'(t)}\right)' e^{i\psi(t)x}dt.$$

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$$f(a)e^{ix\psi(a)}\int_a^{a+\epsilon} \exp(ix\psi^{(p)(a)}(t-a)^p/p!])dt$$

Integral decays faster and faster as $x \to \infty$, so to leading order it is

$$\sim \int_a^{\infty} \exp(ix\psi^{(p)}(t-a)^p/p])dt = e^{\pm i\pi/2p}\left[\frac{p!}{x|\psi^{(p)}(a)|}\right]^{1/p}\Gamma(1/p)/p.$$

where $+$ is chosen for $\psi^{(p)}(a) > 0$ etc.

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Remark: in computing integrals $\int_0^{\infty} e^{ias^p}ds$, use substitution $s = e^{\pm i\pi/2p}(u/|a|)^{1/p}$.

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The stationary phase point is $t = 1/2$. Expanding $t - t^2 = 1/4 - (t-1/2)^2$ and $1/(1+t^2) \approx 4/5$, the leading order estimate is

$$\frac{4}{5} \int_{\infty}^{\infty} \exp(ix/4) exp(-ix(t-1/2)^2)\, dt = \frac{4}{5} \exp(ix/4) e^{-i\pi/4} \sqrt{\pi/x}.$$

Asymptotics the Bessel integral

Consider

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin t - nt)\, dt = \mathrm{Re}\int_0^{\pi} e^{-int} e^{ix\sin t}\, dt.$$

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For $x \to 0$, this is not a stationary phase integral, but the exponential can be expanded directly:

$$J_n \sim \mathrm{Re} \frac{1}{\pi}\int_0^{\pi} e^{-int} \sum_{k=0}^{\infty} (ix \sin t)^k / k!\, dt.$$

Note $(i \sin t)^k = (1/2)^k (e^{ikt} + \ldots + e^{-ikt})$. By orthogonality, the first nonzero term in the sum is where $k = n$, thus

$$J_n \sim \frac{x^n}{\pi n!} \int_0^{\pi} \frac{1}{2^n} e^{-int} e^{int} dt = (x/2)^n / n!, \quad x \to 0.$$

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The other limit $x \to \infty$ has a stationary phase points where $\cos t = 0$ or $t = \pi/2$. Thus

$$J_n \sim \frac{1}{\pi}\mathrm{Re}\int_{\infty}^{\infty} e^{-in\pi/2}e^{ix(1-(t-\pi/2)^2/2}\,dt = \sqrt{2/(\pi x)}\cos(x - n\pi/2 - \pi/4).$$

Integrands that vanish at stationary phase point

What if $f(0) = 0$ for integral $\int f(t)e^{ix\psi(t)}\,dt$ with stationary phase point $t = 0$? Take $f(t) \sim at + bt^2$ for $t \to 0$, and $f > 0$ bounded, and consider

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Trick: integrate by parts first, which brings integral into standard form.

$$\begin{aligned} I(x) &= \int_0^\infty \left(\frac{f(t)}{t}\right) te^{ixt^2}\,dt \\ &= \frac{e^{ixt^2}}{2ix}\frac{f(t)}{t}\bigg|_0^\infty - \frac{1}{2ix}\int_0^\infty \left(\frac{f(t)}{t}\right)' e^{ixt^2}\,dt \\ &= -\frac{a}{2ix} - \frac{b}{2ix}\int_0^\infty e^{ixt^2}\,dt \\ &= -\frac{a}{2ix} - \frac{b}{2ix}\left(\frac{\pi}{x^3}\right)^{1/2} e^{i\pi 4}. \end{aligned}$$