Asymptotic Methods

Method of stationary phase
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More generally, if \( I = \int_a^b f(t)e^{ix\psi(t)} \, dt \), then integration by parts can be used to obtain
\[ \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \bigg|_{t=a}^{t=b} - \frac{1}{ix\psi'(t)} \int_a^b \left( \frac{f(t)}{\psi'(t)} \right)' e^{ix\psi(t)} \, dt. \]

Provided \( \psi' \) is never zero, IBP can be repeated, and \( I \sim 1/x \).
Highly oscillatory integrals

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But what if $$\psi'(x) = 0$$? Suppose that $$\psi \sim \psi(a) + \psi^{(p)}(a)(t - a)^p/p!$$ with $$p \geq 2$$. Split integral $$\int_{a}^{a+\epsilon} + \int_{a+\epsilon}^{b}$$, so that the latter integral is $$O(1/x)$$. 

Remark: in computing integrals $$\int_{0}^{\infty} e^{ias} ds$$, use substitution $$s = e^{\pm i\pi/2} (u/|a|)^{1/p}$$. 

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The former integral can be estimated by inserting expansion for $\psi$

$$f(a)e^{ix\psi(a)} \int_{a}^{a+\epsilon} \exp(ix\psi^{(p)}(a)(t - a)^p / p!) dt$$

Integral decays faster and faster as $x \to \infty$, so to leading order it is

$$\sim \int_{a}^{\infty} \exp(ix\psi^{(p)}(t - a)^p / p]) dt = e^{\pm i\pi/2p} \left[ \frac{p!}{x|\psi^{(p)}(a)|} \right]^{1/p} \Gamma(1/p)/p.$$ 

where $+$ is chosen for $\psi^{(p)}(a) > 0$ etc.
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Remark: in computing integrals \( \int_0^{\infty} e^{ias^p} \, ds \), use substitution \( s = e^{\pm i\pi/2p(u/|a|)} \).
The points where $\phi'(x) = 0$ are called stationary phase points. Just like Laplace points, the integral’s main contribution can be approximated by expanding around them.
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The stationary phase point is $t = 1/2$. Expanding $t - t^2 = 1/4 - (t - 1/2)^2$ and $1/(1 + t^2) \approx 4/5$, the leading order estimate is

$$\frac{4}{5} \int_{-\infty}^{\infty} \exp(i x/4) \exp(-i x(t - 1/2)^2) dt = \frac{4}{5} \exp(i x/4) e^{-i \pi/4} \sqrt{\pi/x}.$$
Asymptotics the Bessel integral

Consider

\[ J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin t - nt) \, dt = \text{Re} \int_0^{\pi} e^{-int} e^{ix \sin t} \, dt. \]
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For \( x \to 0 \), this is not a stationary phase integral, but the exponential can be expanded directly:

\[ J_n \sim \text{Re} \frac{1}{\pi} \int_0^{\pi} e^{-int} \sum_{k=0}^{\infty} \frac{(ix \sin t)^k}{k!} \, dt. \]

Note \((i \sin t)^k = (1/2)^k (e^{ikt} + \ldots + e^{-ikt})\). By orthogonality, the first nonzero term in the sum is where \( k = n \), thus

\[ J_n \sim \frac{x^n}{\pi n!} \int_0^{\pi} \frac{1}{2^n} e^{-int} e^{int} \, dt = \frac{(x/2)^n}{n!}, \quad x \to 0. \]
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The other limit \( x \to \infty \) has a stationary phase points where \( \cos t = 0 \) or \( t = \pi/2 \). Thus
\[ J_n \sim \frac{1}{\pi} \text{Re} \int_{-\infty}^{\infty} e^{-in\pi/2} e^{ix(1-(t-\pi/2)^2)/2} \, dt = \sqrt{2/(\pi x)} \cos(x - n\pi/2 - \pi/4). \]
What if \( f(0) = 0 \) for integral \( \int f(t)e^{ix\psi(t)} \, dt \) with stationary phase point \( t = 0 \)?

Take \( f(t) \sim at + bt^2 \) for \( t \to 0 \), and \( f > 0 \) bounded, and consider

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I(x) = \int_0^\infty f(t)e^{ixt^2} \, dt
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Integrands that vanish at stationary phase point

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Cannot just use leading order expansion for $f()$, since

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Trick: integrate by parts first, which brings integral into standard form.

\[
I(x) = \int_0^\infty \left( \frac{f(t)}{t} \right) te^{ixt^2} \, dt
\]

\[
= \frac{e^{ixt^2}}{2ix} \left. \frac{f(t)}{t} \right|_0^\infty - \frac{1}{2ix} \int_0^\infty \left( \frac{f(t)}{t} \right)' e^{ixt^2} \, dt
\]

\[
= -\frac{a}{2ix} - \frac{b}{2ix} \int_0^\infty e^{ixt^2} \, dt
\]

\[
= -\frac{a}{2ix} - \frac{b}{2ix} \left( \frac{\pi}{x^3} \right)^{1/2} e^{i\pi^4}.
\]