Asymptotic Methods

The method of steepest descent
Main observation: oscillatory integrals can often be turned into exponentially decaying ones by deforming the complex contour, e.g.

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To make this work in practice for integrals of the form

\[ \int_C f(t)e^{x\rho(t)} \, dt, \]

need to find contours for which \(|e^{\rho(t)}|\) increases (or decreases) the fastest; these are steepest descent contours.
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- Note that $|e^{\rho(t)}| = e^{\text{Re} (\rho(t))}$. We therefore want contours orthogonal to the level sets of $\text{Re} (\rho(t))$; by the Cauchy-Riemann equations, these are generally the level sets of $\text{Im} (\rho(t))$. 

Steepest descent integrals

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- Usually, \( |e^{\rho(t)}| \) will be largest on certain points on the steepest descent contour, and will decay exponentially away from these points. In other words, we have transformed the problem into a Laplace integral.

- The Laplace points are generally where \( \rho'(t) = 0 \); these are saddle points. Often, simply identifying the saddle points (instead of the whole contour) is sufficient.
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\[ i \cosh t = -\sinh u \sin \nu + i \cosh u \cos \nu, \quad t = u + iv, \]

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One way to parameterize this curve is to use the real part $\phi = \sinh u \sin v$ as the parameter. With this choice, the deformed integral can be written

$$2 \int_{0}^{\infty} \frac{e^{-\phi x} e^{ix}}{i \sqrt{\phi} (2i - \phi)^{1/2}} d\phi,$$

where the principal square root is assumed. A binomial expansion of the denominator produces

$$\frac{2}{\sqrt{2i}} e^{ix} \int_{0}^{\infty} e^{-\phi x} \left( \frac{1}{\sqrt{\phi}} - \frac{i \sqrt{\phi}}{4} + \ldots \right) d\phi$$

$$= \sqrt{2} e^{i\pi/4} e^{ix} \left( \Gamma(1/2)x^{-1/2} - \frac{i \Gamma(3/2)}{4} x^{-3/2} + \ldots \right)$$
Consider

\[ I(x) = \int_0^1 \ln t \ e^{ixt} \ dt, \ x \to \infty \]

which is not of stationary phase type. The steepest descent curves are where \( \text{Re } t \) is a constant. Deform contour to a union of three straight line contours:
Taking $C_2$ to infinity, we are left with

$$
\int_{C_1} = i \int_0^\infty \ln(is) e^{-xs} \, ds = \frac{-i \ln x - i\gamma + \pi/2}{x}
$$

and

$$
\int_{C_3} = -i \int_0^\infty \ln(1 + is) e^{ix} e^{-xs} \, ds
$$

$$
= (-i) e^{ix} \int_0^\infty (is + s^2/2 - is^3/3 + \ldots) e^{-xs} \, ds
$$

$$
= -ie^{ix} (i/x^2 + 1/x^3 + \ldots)
$$
For integrals

\[ I = \int_C f(t)e^{x\rho(t)} dt, \quad x \to \infty, \]

suppose that \( C \) can be deformed through a saddle where \( \rho'(t_0) = 0 \) and \( \rho''(t_0) \neq 0 \), so that \( \text{Re} \rho(t_0) \) is a maximum along the contour.
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The leading order approximation is entirely dictated by the integrand near the saddle, and therefore it suffices to write
\[ I \sim \int_C f(t_0) e^{x(\rho(t_0) + \rho''(t_0)(t-t_0)/2)} dt, \quad x \to \infty. \]

Therefore we do not need to know the steepest descent contour exactly. For this approximation, the contour is a line that can be parameterized \( t = t_0 + se^{i\theta} \), where \( \theta \) is chosen so that the exponential will decay and not oscillate, i.e.
\[ \arg(\rho''(t_0)e^{2i\theta}) = \pi, \quad \theta = (\pi - \arg(\rho''(t_0))/2. \]
Consider

\[ I(x) = \int_{-\infty}^{\infty} e^{-xt^2} \cos(xt) f(t) \, dt = \text{Re} \int_{-\infty}^{\infty} e^{-x(t^2-it)} f(t) \, dt, \quad x \to \infty. \]
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The saddle is where \( \frac{d}{dt}(t^2 - it) = 0 \) or \( t_0 = i/2 \), and \( \frac{d^2}{dt^2}(t^2 - it) = -2 \). The steepest descent contour is then approximately parameterized by \( t = i/2 + se^{i\theta} \) where \( \theta = 0 \), leading to

\[ I(x) \sim \text{Re} \int_{-\infty}^{\infty} e^{-x(1/4 + s^2)} f(i/2) dt = \text{Re} f(i/2) e^{-x/4} \sqrt{\frac{\pi}{x}}. \]
The Airy function

The Airy differential equation is

\[ y'' - xy = 0, \quad -\infty < x < \infty. \]

Its decaying solution by Fourier transform is

\[ y(x) \equiv \text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(kx+k^3/3)} dk. \]
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To put this into a form suitable for steepest descents if \( x > 0 \), let \( k = x^{1/2} z \) so that \( \text{Ai}(x) = x^{1/2} I(x^{3/2})/(2\pi) \) with

\[ I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(z+z^3/3)} \, dz. \]
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The saddle points are where \( d/dz(z + z^3/3) = 0 \) or \( z = \pm i \). Which one to use?
Choosing the contour in the upper half plane through \( z = i \), one uses the approximate contour \( z = i + s \). With \( i(z + z^3/3) \approx -2/3 - (z - i)^2 \), \( z \to i \), we get

\[
I \sim \int_{-\infty}^{\infty} e^{-\lambda(-2/3+s^2)} ds = e^{-2\lambda/3} \sqrt{\frac{\pi}{2\lambda}},
\]

so that

\[
\text{Ai}(x) \sim \frac{1}{2\sqrt{\pi} x^{1/4}} e^{-2x^{3/2}/3}.
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For $x \to -\infty$ instead, use $k = |x|^{1/2}$ so $\text{Ai}(x) = |x|^{1/2} I(|x|^{3/2})/(2\pi)$ with

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In this case, there are saddles through \( \pm 1 \), and each contributes equally.
With \( i(z - z^3/3) \approx 2/3 i - i(z - i)^2 \), \( z \to 1 \), we use approximate contour \( z = 1 + se^{-i\pi/4} \), and the contribution near this saddle is
\[
e^{-i\pi/4} \int_{-\infty}^{\infty} e^{\lambda(2i/3-s^2)} \, ds = e^{-i\pi/4 + 2\lambda i/3} \sqrt{\pi/\lambda}.
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e^{-i\pi/4} \int_{-\infty}^{\infty} e^{\lambda(2i/3-s^2)} \, ds = e^{-i\pi/4+2\lambda i/3} \sqrt{\pi/\lambda}.
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Similarly, \(i(z - z^3/3) \approx -2/3i + i(z - i)^2\), \(z \to -1\), and using the approximate contour \(z = -1 + se^{i\pi/4}\), the contribution near this saddle is

\[
e^{i\pi/4} \int_{-\infty}^{\infty} e^{\lambda(-2i/3-s^2)} \, ds = e^{i\pi/4-2\lambda i/3} \sqrt{\pi/\lambda}.
\]

Adding both contributions, one finally gets

\[
\text{Ai}(x) \sim \frac{1}{\sqrt{\pi x^{1/4}}} \cos\left(2x^{3/2}/3 - \pi/4\right).
\]
Dealing with endpoints

Even if contour is deformed through saddle point, the size of the endpoint contribution must be checked. Consider

\[
\int_{-1}^{1} e^{-4xt^2} \cos(5xt - xt^3) \, dt = e^{-2x} \Re \int_{-1}^{1} e^{x \rho(t)} \, dt, \quad \rho(t) = -(t-i)^2 - i(t-i)^3.
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Check: \( \text{Re} \rho(\pm 1) = -2 \), whereas \( \text{Re} \rho(i) = 0 \). Thus saddle contributes exponentially more than endpoints.

Using approximate contour \( t = i + s \), \( \rho(t) \sim -s^2 \) as \( t \to i \), and integral approximates

\[
e^{-2x} \int_{-\infty}^{\infty} e^{-xs^2} \, ds = e^{-2x} \sqrt{\pi/x}. 
\]
Asymptotics of binomial coefficients

Consider

\[
\binom{N}{M} = \frac{1}{2\pi i} \int_C \frac{(1+z)^N}{z^{M+1}} \, dz,
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where \( C \) is a contour around the origin. Want limit where \( M/N = \mu < 1 \) as \( N \to \infty \); write integral as \( \frac{1}{2\pi i} \int_C \frac{1}{z} \exp(N\rho(z)) \, dz \), \( \rho(z) \equiv \ln(1 + z) - \mu \ln z \).

There is a saddle point at \( z_0 = \mu/(1 - \mu) > 0 \), and

\[ \rho(z_0) = -\mu \ln \mu - (1 - \mu) \ln(1 - \mu), \quad \rho''(z_0) = (1 - \mu)^3/\mu. \]
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\]

For \( \mu = 1/2 \), SD contour is just unit circle. Approximate contour is \( z = se^{i\pi/2} + z_0 \).

Integral approximates

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{z_0} e^{N\rho(z_0) - N\rho''(z_0)s^2} e^{i\pi/2} \, ds = \left[ \frac{1}{2\pi N\mu(1-\mu)} \right]^{1/2} \exp(N\rho(z_0)).
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Higher order saddles

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$$\int_{-\infty}^{\infty} e^{x[\cosh(t-i\pi)-(t-i\pi)^2/2]} dt.$$  

Near saddle at $t = i\pi$, $\rho(t) \approx 1 + (t - i\pi)^4/4!$.

Contour to left of saddle uses approximate parameterization $t = i\pi + se^{i\pi/4}$, $-\infty < s < 0$, whereas to the right it is $t = i\pi + se^{-i\pi/4}$, $0 < s < \infty$. Resulting approximation is

$$\int_{0}^{\infty} e^{-i\pi/4} e^{x(1-r^4/24)} dr + c.c. = \frac{1}{2} e^{x(6/x)^{1/4}} \gamma(1/4).$$
Consider

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Looking for traveling wave \( u = g(x - ct) \) produces ODE

\[ g'' + cg' + g(1 - g) = 0, \quad g(-\infty) = 1, \quad g(\infty) = 0, \]

Phase plane reveals that any \( c > 0 \) gives a solution, EXCEPT that if \( c < \sqrt{2} \), then front is not monotone. What happens if initial condition is positive and decays rapidly as \( x \to \infty \)?
Hypothesis: front is “pulled along” by behavior in tail where $u \to 0$, which satisfies the linear equation

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The Fourier solution is

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u(x,t) = \int_{-\infty}^{\infty} e^{ikx} e^{(1-k^2)t} \hat{f}(k) dk,
\]

so that running along side the wave at speed \( c \) gives

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u(ct,t) = \int_{-\infty}^{\infty} e^{(1-k^2+ik^2)t} \hat{f}(k) dk.
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$$u(ct, t) = \int_{-\infty}^{\infty} e^{(1-k^2+ick)t} \hat{f}(k) dk.$$

For $t \to \infty$, this is a steepest descents problem with saddle at $k^* = -ic/2$, and the resulting approximation to the integral is

$$u(ct, t) \sim e^{t(1-c^2/2)} \int_{-\infty}^{\infty} e^{-(k-k^*)^2} \hat{f}(k^*) dk.$$
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“Marginal stability” property: choose $c$ so that waves amplitude does not grow or shrink, so that $1 - c^2/2 = 0$ or $c = \sqrt{2}$. 