Asymptotic Methods

The method of steepest descent

Steepest descent integrals

Main observation: oscillatory integrals can often be turned into exponentially decaying ones by deforming the complex contour, e.g.

$$\int_0^\infty e^{ix^2}\,dx = e^{i\pi/4}\int_0^\infty e^{-u^2}\,du, \quad x = e^{i\pi/4}u.$$

Steepest descent integrals

Main observation: oscillatory integrals can often be turned into exponentially decaying ones by deforming the complex contour, e.g.

$$\int_0^\infty e^{ix^2}dx = e^{i\pi/4}\int_0^\infty e^{-u^2}du, \quad x = e^{i\pi/4}u.$$

To make this work in practice for integrals of the form

$$\int_C f(t)e^{x\rho(t)}dt,$$

need to find contours for which $|e^{\rho(t)}|$ increases (or decreases) the fastest; these are steepest descent contours.

Steepest descent integrals

Main observation: oscillatory integrals can often be turned into exponentially decaying ones by deforming the complex contour, e.g.

$$\int_0^\infty e^{ix^2}\,dx = e^{i\pi/4}\int_0^\infty e^{-u^2}\,du, \quad x = e^{i\pi/4}u.$$

To make this work in practice for integrals of the form

$$\int_C f(t)e^{x\rho(t)}\,dt,$$

need to find contours for which $|e^{\rho(t)}|$ increases (or decreases) the fastest; these are steepest descent contours.

- Note that $|e^{\rho(t)}| = e^{\mathrm{Re}\,(\rho(t))}$. We therefore want contours orthogonal to the level sets of Re $(\rho(t))$; by the Cauchy-Riemann equations, these are generally the level sets of Im $(\rho(t))$.

Steepest descent integrals

Main observation: oscillatory integrals can often be turned into exponentially decaying ones by deforming the complex contour, e.g.

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \int_0^\infty e^{-u^2} du, \quad x = e^{i\pi/4} u.$$

To make this work in practice for integrals of the form

$$\int_C f(t) e^{x\rho(t)} dt,$$

need to find contours for which $|e^{\rho(t)}|$ increases (or decreases) the fastest; these are steepest descent contours.

- Note that $|e^{\rho(t)}| = e^{\mathrm{Re}\,(\rho(t))}$. We therefore want contours orthogonal to the level sets of Re $(\rho(t))$; by the Cauchy-Riemann equations, these are generally the level sets of Im $(\rho(t))$.
- Usually, $|e^{\rho(t)}|$ will be largest on certain points on the steepest descent contour, and will decay exponentially away from these points. In other words, we have transformed the problem into a Laplace integral.

Steepest descent integrals

Main observation: oscillatory integrals can often be turned into exponentially decaying ones by deforming the complex contour, e.g.

$$\int_0^\infty e^{ix^2}\,dx = e^{i\pi/4}\int_0^\infty e^{-u^2}\,du, \quad x = e^{i\pi/4}u.$$

To make this work in practice for integrals of the form

$$\int_C f(t)e^{x\rho(t)}\,dt,$$

need to find contours for which $|e^{\rho(t)}|$ increases (or decreases) the fastest; these are steepest descent contours.

- Note that $|e^{\rho(t)}| = e^{\text{Re }(\rho(t))}$. We therefore want contours orthogonal to the level sets of Re $(\rho(t))$; by the Cauchy-Riemann equations, these are generally the level sets of Im $(\rho(t))$.
- Usually, $|e^{\rho(t)}|$ will be largest on certain points on the steepest descent contour, and will decay exponentially away from these points. In other words, we have transformed the problem into a Laplace integral.
- The Laplace points are generally where $\rho'(t) = 0$; these are saddle points. Often, simply identifying the saddle points (instead of the whole contour) is sufficient.

Example

Consider

$$\int_{-\infty}^{\infty} e^{ix \cosh t} dt, \quad x \to \infty$$

Only the leading order term can be computed with stationary phase.

Example

Consider

$$\int_{-\infty}^{\infty} e^{ix \cosh t} dt, \quad x \to \infty$$

Only the leading order term can be computed with stationary phase. Writing

$$i \cosh t = -\sinh u \sin v + i \cosh u \cos v, \quad t = u + iv,$$

the steepest descent contours are level sets of $\cosh u \cos v$; symmetry suggests a contour through the origin $\cosh u \cos v = 1$ is the correct choice.

Example

Consider

$$\int_{-\infty}^{\infty} e^{ix\cosh t}dt, \quad x \to \infty$$

Only the leading order term can be computed with stationary phase. Writing

$$i\cosh t = -\sinh u \sin v + i\cosh u \cos v, \quad t = u + iv,$$

the steepest descent contours are level sets of $\cosh u \cos v$; symmetry suggests a contour through the origin $\cosh u \cos v = 1$ is the correct choice.

One way to parameterize this curve is to use the real part $\phi = \sinh u \sin v$ as the parameter. With this choice, the deformed integral can be written

$$2\int_0^{\infty} \frac{e^{-\phi x}e^{ix}}{i\sqrt{\phi}(2i-\phi)^{1/2}}d\phi,$$

where the principal square root is assumed. A binomial expansion of the denominator produces

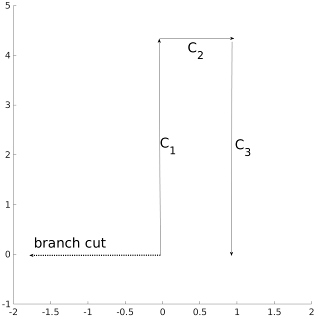
$$\frac{2}{\sqrt{2i}}e^{ix}\int_0^{\infty} e^{-\phi x}\left(\frac{1}{\sqrt{\phi}} - \frac{i\sqrt{\phi}}{4} + \dots\right)d\phi$$
$$= \sqrt{2}e^{i\pi/4}e^{ix}\left(\Gamma(1/2)x^{-1/2} - \frac{i\Gamma(3/2)}{4}x^{-3/2} + \dots\right)$$

Example

Consider

$$I(x) = \int_0^1 \ln t \, e^{ixt} dt, x \to \infty$$

which is not of stationary phase type. The steepest descent curves are where Re t is a constant. Deform contour to a union of three straight line contours:



Example, cont.

Taking C_2 to infinity, we are left with

$$\int_{C_1} = i\int_0^\infty \ln(is)e^{-xs}ds = \frac{-i\ln x - i\gamma + \pi/2}{x}$$

and

$$\begin{aligned}\int_{C_3} &= -i\int_0^\infty \ln(1+is)e^{ix}e^{-xs}ds \\ &= (-i)e^{ix}\int_0^\infty (is + s^2/2 - is^3/3 + \ldots)e^{-xs}ds \\ &= -ie^{ix}(i/x^2 + 1/x^3 + \ldots)\end{aligned}$$

Use of approximate contours near a saddle point

For integrals

$$I = \int_C f(t) e^{x\rho(t)} dt, \quad x \to \infty,$$

suppose that C can be deformed through a saddle where $\rho'(t_0) = 0$ and $\rho''(t_0) \neq 0$, so that Re $\rho(t_0)$ is a maximum along the contour.

Use of approximate contours near a saddle point

For integrals

$$I = \int_C f(t) e^{x\rho(t)} dt, \quad x \to \infty,$$

suppose that C can be deformed through a saddle where $\rho'(t_0) = 0$ and $\rho''(t_0) \neq 0$, so that Re $\rho(t_0)$ is a maximum along the contour.

The leading order approximation is entirely dictated by the integrand near the saddle, and therefore it suffices to write

$$I \sim \int_C f(t_0) e^{x(\rho(t_0) + \rho''(t_0)(t-t_0)/2} dt, \quad x \to \infty.$$

Therefore we do not need to know the steepest descent contour exactly. For this approximation, the contour is a line that can be parameterized $t = t_0 + se^{i\theta}$, where θ is chosen so that the exponential will decay and not oscillate, i.e.

$$\arg(\rho''(t_0)e^{2i\theta}) = \pi, \quad \theta = (\pi - \arg(\rho''(t_0))/2.$$

Example

Consider

$$I(x) = \int_{-\infty}^{\infty} e^{-xt^2} \cos(xt) f(t) dt = \mathrm{Re} \int_{-\infty}^{\infty} e^{-x(t^2 - it)} f(t) dt, \quad x \to \infty.$$

Example

Consider

$$I(x) = \int_{-\infty}^{\infty} e^{-xt^2} \cos(xt) f(t) dt = \text{Re} \int_{-\infty}^{\infty} e^{-x(t^2 - it)} f(t) dt, \quad x \to \infty.$$

The saddle is where $d/dt(t^2 - it) = 0$ or $t_0 = i/2$, and $d^2/dt^2(t^2 - it) = -2$. The steepest descent contour is then approximately parameterized by $t = i/2 + se^{i\theta}$ where $\theta = 0$, leading to

$$I(x) \sim \text{Re} \int_{-\infty}^{\infty} e^{-x(1/4+s^2)} f(i/2) dt = \text{Re}\ f(i/2) e^{-x/4} \sqrt{\frac{\pi}{x}}.$$

The Airy function

The Airy differential equation is

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Its decaying solution by Fourier transform is

$$y(x) \equiv \mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(kx + k^3/3)} dk.$$

The Airy function

The Airy differential equation is

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Its decaying solution by Fourier transform is

$$y(x) \equiv \mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(kx+k^3/3)}dk.$$

To put this into a form suitable for steepest descents if $x > 0$, let $k = x^{1/2}z$ so that $\mathrm{Ai}(x) = x^{1/2}I(x^{3/2})/(2\pi)$ with

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(z+z^3/3)}dz.$$

The Airy function

The Airy differential equation is

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Its decaying solution by Fourier transform is

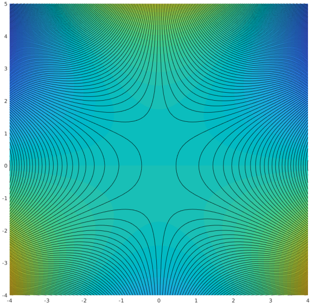
$$y(x) \equiv \mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(kx + k^3/3)} dk.$$

To put this into a form suitable for steepest descents if $x > 0$, let $k = x^{1/2} z$ so that $\mathrm{Ai}(x) = x^{1/2} I(x^{3/2})/(2\pi)$ with

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(z + z^3/3)} dz.$$

The saddle points are where $d/dz(z + z^3/3) = 0$ or $z = \pm i$. Which one to use?

The Airy function, cont.



Choosing the contour in the upper half plane through $z = i$, one uses the approximate contour $z = i + s$. With $i(z + z^3/3) \approx -2/3 - (z - i)^2$, $z \to i$, we get

$$I \sim \int_{-\infty}^{\infty} e^{-\lambda(-2/3+s^2} ds = e^{-2\lambda/3}\sqrt{\frac{\pi}{2\lambda}},$$

so that

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-2x^{3/2}/3}.$$

The Airy function, cont.

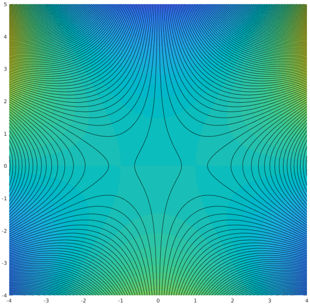
For $x \to -\infty$ instead, use $k = |x|^{1/2} z$ so $\text{Ai}(x) = |x|^{1/2} I(|x|^{3/2})/(2\pi)$ with

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(z - z^3/3)} dz.$$

The Airy function, cont.

For $x \to -\infty$ instead, use $k = |x|^{1/2} z$ so $\text{Ai}(x) = |x|^{1/2} I(|x|^{3/2})/(2\pi)$ with

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(z - z^3/3)} dz.$$



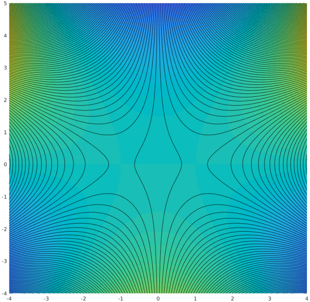
In this case, there are saddles through ± 1, and each contributes equally.
With $i(z - z^3/3) \approx 2/3i - i(z-i)^2$, $z \to 1$, we use approximate contour $z = 1 + se^{-i\pi/4}$, and the contribution near this saddle is

$$e^{-i\pi/4} \int_{-\infty}^{\infty} e^{\lambda(2i/3 - s^2)} ds = e^{-i\pi/4 + 2\lambda i/3} \sqrt{\pi/\lambda}.$$

The Airy function, cont.

For $x \to -\infty$ instead, use $k = |x|^{1/2} z$ so $\text{Ai}(x) = |x|^{1/2} I(|x|^{3/2})/(2\pi)$ with

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda(z - z^3/3)} dz.$$



In this case, there are saddles through ± 1, and each contributes equally.
With $i(z - z^3/3) \approx 2/3i - i(z - i)^2$, $z \to 1$, we use approximate contour $z = 1 + se^{-i\pi/4}$, and the contribution near this saddle is

$$e^{-i\pi/4} \int_{-\infty}^{\infty} e^{\lambda(2i/3 - s^2)} ds = e^{-i\pi/4 + 2\lambda i/3} \sqrt{\pi/\lambda}.$$

Similarly, $i(z - z^3/3) \approx -2/3i + i(z - i)^2$, $z \to -1$, and using the approximate contour $z = -1 + se^{i\pi/4}$, the contribution near this saddle is

$$e^{i\pi/4} \int_{-\infty}^{\infty} e^{\lambda(-2i/3 - s^2)} ds = e^{i\pi/4 - 2\lambda i/3} \sqrt{\pi/\lambda}.$$

Adding both contributions, one finally gets

$$\text{Ai}(x) \sim \frac{1}{\sqrt{\pi} x^{1/4}} \cos(2x^{3/2}/3 - \pi/4).$$

Dealing with endpoints

Even if contour is deformed through saddle point, the size of the endpoint contribution must be checked. Consider

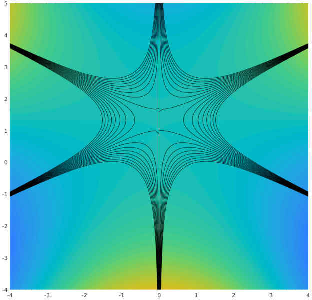
$$\int_{-1}^{1} e^{-4xt^2}\cos(5xt - xt^3)\,dt = e^{-2x}\mathrm{Re}\int_{-1}^{1} e^{x\rho(t)}\,dt, \quad \rho(t) = -(t-i)^2 - i(t-i)^3.$$

Dealing with endpoints

Even if contour is deformed through saddle point, the size of the endpoint contribution must be checked. Consider

$$\int_{-1}^{1} e^{-4xt^2}\cos(5xt - xt^3)\,dt = e^{-2x}\,\mathrm{Re}\int_{-1}^{1} e^{x\rho(t)}\,dt, \quad \rho(t) = -(t-i)^2 - i(t-i)^3.$$

There is clearly a saddle at $t = i$, but how to deform contour?

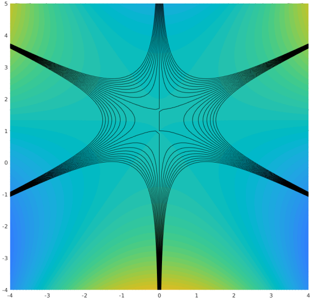


Dealing with endpoints

Even if contour is deformed through saddle point, the size of the endpoint contribution must be checked. Consider

$$\int_{-1}^{1} e^{-4xt^2}\cos(5xt - xt^3)dt = e^{-2x}\text{Re}\int_{-1}^{1} e^{x\rho(t)}dt, \quad \rho(t) = -(t-i)^2 - i(t-i)^3.$$

There is clearly a saddle at $t = i$, but how to deform contour?



Check: Re $\rho(\pm 1) = -2$, whereas Re $\rho(i) = 0$. Thus saddle contributes exponentially more than endpoints.

Using approximate contour $t = i + s$, $\rho(t) \sim -s^2$ as $t \to i$, and integral approximates

$$e^{-2x}\int_{-\infty}^{\infty} e^{-xs^2}ds = e^{-2x}\sqrt{\pi/x}.$$

Asymptotics of binomial coefficients

Consider

$$\binom{N}{M} = \frac{1}{2\pi i} \int_C \frac{(1+z)^N}{z^{M+1}} dz,$$

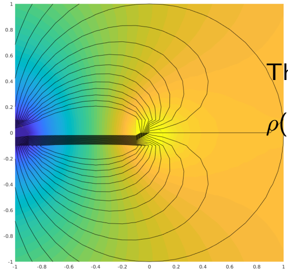
where C is a contour around the origin.

Asymptotics of binomial coefficients

Consider

$$\binom{N}{M} = \frac{1}{2\pi i}\int_C \frac{(1+z)^N}{z^{M+1}}\,dz,$$

where C is a contour around the origin. Want limit where $M/N = \mu < 1$ as $N \to \infty$; write integral as $\frac{1}{2\pi i}\int_C \frac{1}{z}\exp(N\rho(z))dz, \quad \rho(z) \equiv \ln(1+z) - \mu \ln z.$



There is a saddle point at $z_0 = \mu/(1-\mu) > 0$, and

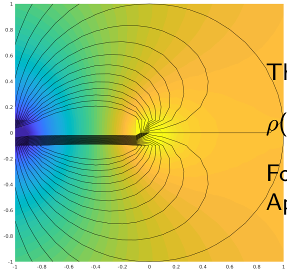
$\rho(z_0) = -\mu \ln \mu - (1-\mu)\ln(1-\mu)$, $\rho''(z_0) = (1-\mu)^3/\mu$.

Asymptotics of binomial coefficients

Consider

$$\binom{N}{M} = \frac{1}{2\pi i}\int_C \frac{(1+z)^N}{z^{M+1}}dz,$$

where C is a contour around the origin. Want limit where $M/N = \mu < 1$ as $N \to \infty$; write integral as $\frac{1}{2\pi i}\int_C \frac{1}{z}\exp(N\rho(z))dz, \quad \rho(z) \equiv \ln(1+z) - \mu \ln z$.



There is a saddle point at $z_0 = \mu/(1-\mu) > 0$, and

$\rho(z_0) = -\mu \ln \mu - (1-\mu)\ln(1-\mu)$, $\rho''(z_0) = (1-\mu)^3/\mu$.

For $\mu = 1/2$, SD contour is just unit circle.
Approximate contour is $z = se^{i\pi/2} + z_0$.

Integral approximates

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{z_0} e^{N\rho(z_0) - N\rho''(z_0)s^2} e^{i\pi/2} ds = \left[\frac{1}{2\pi N\mu(1-\mu)}\right]^{1/2} exp(N\rho(z_0)).$$

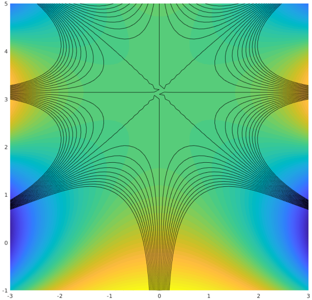
Higher order saddles

When $\rho'' = 0$ at saddle, must investigate how saddle should be approached.

Higher order saddles

When $\rho'' = 0$ at saddle, must investigate how saddle should be approached. Consider

$$\int_{-\infty}^{\infty} e^{x[\cosh(t-i\pi)-(t-i\pi)^2/2]}dt.$$



Near saddle at $t = i\pi$, $\rho(t) \approx 1 + (t - i\pi)^4/4!$.

Contour to left of saddle uses approximate parameterization $t = i\pi + se^{i\pi/4}, -\infty < s < 0$, whereas to the right it is $t = i\pi + se^{-i\pi/4}, 0 < s < \infty$. Resulting approximation is

$$\int_0^{\infty} e^{-i\pi/4}e^{x(1-r^4/24)}dr + c.c. = \frac{1}{2}e^x(6/x)^{1/4}\gamma(1/4).$$

Front propagation in Fisher's equation

Consider

$$u_t = u_{xx} + u(1-u), \quad u(-\infty, t) = 1, \quad u(+\infty, t) = 0.$$

Front propagation in Fisher's equation

Consider

$$u_t = u_{xx} + u(1-u), \quad u(-\infty, t) = 1, \quad u(+\infty, t) = 0.$$

Looking for traveling wave $u = g(x - ct)$ produces ODE

$$g'' + cg' + g(1-g) = 0, \quad g(-\infty) = 1, \quad g(\infty) = 0,$$

Phase plane reveals that any $c > 0$ gives a solution, EXCEPT that if $c < \sqrt{2}$, then front is not monotone. What happens if initial condition is positive and decays rapidly as $x \to \infty$?

Front propagation in Fisher's equation,cont.

Hypothesis: front is "pulled along" by behavior in tail where $u \to 0$, which satisfies the linear equation

$$u_t = u_{xx} + u, \quad u(x,0) = f(x).$$

Front propagation in Fisher's equation,cont.

Hypothesis: front is "pulled along" by behavior in tail where $u \to 0$, which satisfies the linear equation

$$u_t = u_{xx} + u, \quad u(x,0) = f(x).$$

The Fourier solution is

$$u(x,t) = \int_{-\infty}^{\infty} e^{ikx} e^{(1-k^2)t} \hat{f}(k) dk,$$

so that running along side the wave at speed c gives

$$u(ct,t) = \int_{-\infty}^{\infty} e^{(1-k^2+ick)t} \hat{f}(k) dk.$$

Front propagation in Fisher's equation,cont.

Hypothesis: front is "pulled along" by behavior in tail where $u \to 0$, which satisfies the linear equation

$$u_t = u_{xx} + u, \quad u(x,0) = f(x).$$

The Fourier solution is

$$u(x,t) = \int_{-\infty}^{\infty} e^{ikx} e^{(1-k^2)t} \hat{f}(k) dk,$$

so that running along side the wave at speed c gives

$$u(ct,t) = \int_{-\infty}^{\infty} e^{(1-k^2+ick)t} \hat{f}(k) dk.$$

For $t \to \infty$, this is a steepest descents problem with saddle at $k^* = -ic/2$, and the resulting approximation to the integral is

$$u(ct,t) \sim e^{t(1-c^2/2)} \int_{-\infty}^{\infty} e^{-(k-k^*)^2 t} \hat{f}(k^*) dk.$$

Front propagation in Fisher's equation,cont.

Hypothesis: front is "pulled along" by behavior in tail where $u \to 0$, which satisfies the linear equation

$$u_t = u_{xx} + u, \quad u(x,0) = f(x).$$

The Fourier solution is

$$u(x,t) = \int_{-\infty}^{\infty} e^{ikx} e^{(1-k^2)t} \hat{f}(k) dk,$$

so that running along side the wave at speed c gives

$$u(ct,t) = \int_{-\infty}^{\infty} e^{(1-k^2+ick)t} \hat{f}(k) dk.$$

For $t \to \infty$, this is a steepest descents problem with saddle at $k^* = -ic/2$, and the resulting approximation to the integral is

$$u(ct,t) \sim e^{t(1-c^2/2)} \int_{-\infty}^{\infty} e^{-(k-k^*)^2 t} \hat{f}(k^*) dk.$$

"Marginal stability" property: choose c so that waves amplitude does not grow or shrink, so that $1 - c^2/2 = 0$ or $c = \sqrt{2}$.