Asymptotic Methods

WKB expansion

Failure of multiple scales with slowly varying coefficients

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It seems reasonable to introduce slow scale $X = \epsilon x$, so that

$$y_{xx} + 2\epsilon y_{xX} + \epsilon^2 y_{XX} + q(X)y = 0.$$

Expand $y = y_0(x, X) + \epsilon y_1(x, X) + \ldots$, so that

$$y_{0xx} + q(X)y = 0,$$

whose general solution is $y_0 = a(X)\cos(\sqrt{q(X)}x) + b(X)\sin(\sqrt{q(X)}x)$.

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whose general solution is $y_0 = a(X)\cos(\sqrt{q(X)}x) + b(X)\sin(\sqrt{q(X)}x)$.
The $\mathcal{O}(\epsilon)$ terms are
$$y_{1xx} + y_1 = -2y_{0xX} = (2a'\sqrt{q} + aq'/\sqrt{q})\sin(\sqrt{q}x) + \frac{aq'^2}{2q}x\cos(\sqrt{q}x)$$
$$-(2b'\sqrt{q} + bq'/\sqrt{q})\cos(\sqrt{q}x) - \frac{bq'^2}{2q}x\sin(\sqrt{q}x).$$
While it is possible to eliminate secular terms with $\cos(\sqrt{q}x)$ and $\sin(\sqrt{q}x)$, still left with terms which produce secular growth unless $a, b = 0$.

A better choice of fast scale

Consider instead same problem written in terms of slow variable to begin with

$$\epsilon^2 y'' - q(x)y = 0,$$

If q is constant, solution is $y = ae^{-x\sqrt{q}/\epsilon} + be^{x\sqrt{q}/\epsilon}$.

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If q is constant, solution is $y = ae^{-x\sqrt{q}/\epsilon} + be^{x\sqrt{q}/\epsilon}$.
This motivates ansatz

$$y \sim e^{\theta(x)/\epsilon}(y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2 + \dots),$$

where $\theta(x)$ is a fast scale variable to be determined. Substitution into the equation gives

$$\theta_x^2 y_0 + \epsilon[\theta_{xx} y_0 + 2\theta_x y_{0x} + (\theta_x)^2 y_1] - q(x)(y_0 + \epsilon y_1 + \epsilon^2 y_2) \sim 0.$$

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Leading order problem is $\theta_x^2 = q(x)$ ("Eikonal equation"), so that by direct integration, $\theta = \pm \int^x \sqrt{q(s)} ds$.

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Superposition of leading order solutions:

$$y \sim q(x)^{-1/4}\left[a_0 \exp\left(-\frac{1}{\epsilon}\int^x \sqrt{q(s)}ds\right) + b_0 \exp\left(\frac{1}{\epsilon}\int^x \sqrt{q(s)}ds\right)\right].$$

An eigenvalue problem

Consider

$$y'' + \lambda^2 e^{2x} y = 0, \quad y(0) = 0 = y(1).$$

Setting $\lambda = 1/\epsilon$ puts this in form of previous problem,providing $\lambda \gg 1$.

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Implementing boundary conditions gives

$$a_0 \cos\lambda + b_0 \sin\lambda = 0,$$
$$a_0 \cos(\lambda e) + b_0 \sin(\lambda e) = 0.$$

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Eigenvalues correspond to where this linear system is singular,

$$\cos(\lambda)\sin(e\lambda) - \sin(\lambda)\cos(e\lambda) = \sin(\lambda(e-1)) = 0.$$

It follows that $\lambda_n \sim n\pi/(e-1)$ for $n = 1, 2, 3, \ldots$ and eigenfunctions are

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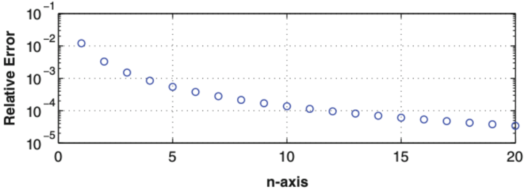
But how good is this approximation?



Turning points and validity of approximation

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To investigate validity of expansion near turning points, continue to $\mathcal{O}(\epsilon^2)$,

$$2\theta_x y_{1x} + \theta_{xx} y_1 = -y_0''.$$

Find particular solution by variation of parameters $y_1 = \omega(x) y_0(x)$, so that

$$\omega = \frac{q_x}{8q^{3/2}} + \frac{1}{32}\int^x \frac{q_x^2(s)}{q^{5/2}} ds.$$

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Near first order turning point x_0, $q \sim C(x - x_0)$ so that $y_1 = \mathcal{O}(q^{-3/2})$ for small q. Expansion will be disordered when this is $\mathcal{O}(1/\epsilon)$ or $q = \mathcal{O}(\epsilon^{2/3})$.

Resolution: seek internal layer solution near turning point which matches outer WKB solution.

Internal layer solution

Suppose that turning point is $x = 0$ and $q'(0) \equiv Q > 0$. Letting $z = x/\epsilon^{\beta}$ be scaled variable, $Y(z) = y(x)$ solves

$$\epsilon^{2-2\beta} Y'' - \epsilon^{\beta} QzY \sim 0$$

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Balance implies $\beta = 2/3$ (as expected). Expand $Y = \epsilon^{\gamma} Y_0(z) + \dots$ so that Y_0 solves Airy equation $Y_{0zz} - QzY_0$, whose solution is

$$Y_0 = a\text{Ai}(Q^{1/3} z) + b\text{Bi}(Q^{1/3} z).$$

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Matching requires behavior

$$\mathrm{Ai}z \sim \begin{cases} \frac{1}{\sqrt{\pi}|z|^{1/4}} \cos(2/3|z|^{3/2} - \pi/4) & z \to -\infty, \\ \frac{1}{2\sqrt{\pi}|z|^{1/4}} \exp(-2/3|z|^{3/2}) & z \to +\infty \end{cases}$$

and

$$\mathrm{Bi}z \sim \begin{cases} \frac{1}{\sqrt{\pi}|z|^{1/4}} \cos(2/3|z|^{3/2} + \pi/4) & z \to -\infty, \\ \frac{1}{\sqrt{\pi}|z|^{1/4}} \exp(2/3|z|^{3/2}) & z \to +\infty. \end{cases}$$

Matching to outer for $x > 0$

Note first that since $q \sim Qx$, $\frac{1}{\epsilon}\int_0^x \sqrt{q(s)}ds \sim \frac{2}{3}Q^{1/2}z^{3/2}$ and $q(x)^{-1/4} \sim \epsilon^{-1/6}Q^{-1/4}z^{-1/4}$ (choose branch $z^{-1/4} = e^{-i\pi/4}|z|^{1/4}$ for $z < 0$).

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The outer solution to the right of the turning point is

$$\begin{aligned} y_R \sim & q(x)^{-1/4}\left[a_R \exp\left(-\frac{1}{\epsilon}\int_0^x \sqrt{q(s)}ds\right) + b_R \exp\left(\frac{1}{\epsilon}\int_0^x \sqrt{q(s)}ds\right)\right] \\ \sim & \epsilon^{-1/6}Q^{-1/4}z^{-1/4}[a_R e^{-2/3Q^{1/2}z^{3/2}} + b_R e^{2/3Q^{1/2}z^{3/2}}]. \end{aligned}$$

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Compare to inner expansion for large $z > 0$,

$$Y \sim \epsilon^{\gamma}\left[\frac{a}{2\sqrt{\pi}Q^{1/12}z^{1/4}}e^{-2/3Q^{1/2}z^{3/2}} + \frac{b}{2\sqrt{\pi}Q^{1/12}z^{1/4}}e^{2/3Q^{1/2}z^{3/2}}\right].$$

It follows that $\gamma = -1/6$, and

$$a_R = \frac{aQ^{1/6}}{2\sqrt{\pi}}, \quad b_R = \frac{bQ^{1/6}}{\sqrt{\pi}}.$$

Matching for $x < 0$

The outer solution to left of the turning point is

$$y_R \sim q(x)^{-1/4}\left[a_L \exp\left(-\frac{1}{\epsilon}\int_0^x \sqrt{q(s)}ds\right) + b_L \exp\left(\frac{1}{\epsilon}\int_0^x \sqrt{q(s)}ds\right)\right]$$

$$\sim \epsilon^{-1/6} Q^{-1/4} |z|^{-1/4} [a_L e^{i(\frac{2}{3}Q^{1/2}z^{3/2}-\pi/4)} + b_L e^{-i(\frac{2}{3}Q^{1/2}z^{3/2}+\pi/4)}].$$

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Compare to inner expansion for large $-z$,

$$Y \sim \frac{\epsilon^{\gamma}}{\sqrt{\pi}|z|^{1/4} Q^{1/12}} \left[a \cos(\frac{2}{3} Q^{1/2} |z|^{3/2} - \pi/4) + b \cos(\frac{2}{3} Q^{1/2} |z|^{3/2} + \pi/4) \right]$$

$$= \frac{\epsilon^{\gamma}}{2\sqrt{\pi}|z|^{1/4} Q^{1/12}} \Big[(a e^{-i\pi/4} + b e^{i\pi/4}) e^{(2i/3) Q^{1/2} z^{3/2}} + (a e^{i\pi/4} + b e^{-i\pi/4}) e^{-(2i/3) Q^{1/2} z^{3/}}$$

It follows that

$$a_L = \frac{Q^{1/6}}{2\sqrt{\pi}} (b + ia), \quad b_L = \frac{Q^{1/6}}{2\sqrt{\pi}} (a + ib).$$

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The outer solution to left of the turning point is

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It follows that

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Upshot: coefficients a_R, a_L, b_R, b_L are linearly related by connection formula

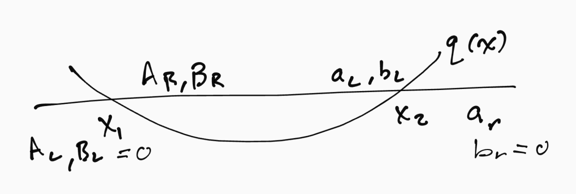
$$\begin{pmatrix} a_L \\ b_L \end{pmatrix} = \begin{pmatrix} i & 1/2 \\ 1 & i/2 \end{pmatrix}\begin{pmatrix} a_R \\ b_R \end{pmatrix}.$$

For $q'(0) < 0$ instead,

$$\begin{pmatrix} a_R \\ b_R \end{pmatrix} = \begin{pmatrix} i/2 & 1 \\ 1/2 & i \end{pmatrix}\begin{pmatrix} a_L \\ b_L \end{pmatrix}.$$

A two-turning point problem

Consider case where $q(x)$ is positive except on interval (x_1, x_2), and boundary conditions are $y(\pm\infty) = 0$.



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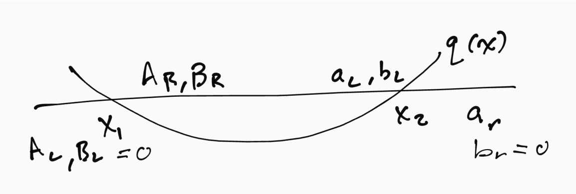

Immediately,

$$y \sim \begin{cases} \frac{A_L}{q^{1/4}} \exp(\frac{1}{\epsilon} \int_{x_1}^{x} \sqrt{q(s)} ds), & B_L = 0 \quad x < x_1, \\ \frac{a_R}{q^{1/4}} \exp(-\frac{1}{\epsilon} \int_{x_2}^{x} \sqrt{q(s)} ds), & b_r = 0 \quad x > x_2. \end{cases}$$

For $x_1 < x < x_2$, have equivalent expressions for y

$$\frac{A_R}{q^{1/4}} \exp\left(\frac{1}{\epsilon} \int_{x_1}^{x} \sqrt{q(s)} ds\right) + \frac{B_R}{q^{1/4}} \exp\left(-\frac{1}{\epsilon} \int_{x_1}^{x} \sqrt{q(s)} ds\right) =$$

$$\frac{a_L}{q^{1/4}} \exp\left(-\frac{1}{\epsilon} \int_{x_2}^{x} \sqrt{q(s)} ds\right) + \frac{b_L}{q^{1/4}} \exp\left(\frac{1}{\epsilon} \int_{x_2}^{x} \sqrt{q(s)} ds\right).$$

It follows that $b_l = IA_R$ and $a_L = B_R/I$ where $I = \exp(\frac{1}{\epsilon} \int_{x_1}^{x_2} \sqrt{q(s)} ds)$.

A two-turning point problem, cont.

Connection formulas imply two independent formulas for a_R,

$$ia_R = a_L = B_R/I = A_L/(2I), \quad a_R = b_L = iA_R = iIA_L/2.$$

Elimination gives $I^2 = -1$ so that

$$\exp\left(\frac{1}{\epsilon}\int_{x_1}^{x_2}\sqrt{q(s)}ds\right) = \exp(i\pi + n\pi i),$$

or

$$\frac{1}{\epsilon}\int_{x_1}^{x_2}\sqrt{-q(s)}ds = (n+\frac{1}{2})\pi, \quad n = 0, 1, 2, \ldots$$

This constrains what $q(x)$ might be for a nontrivial solution to exist.

An eigenvalue problem with two turning points

Consider

$$y'' - |x|y = -\lambda y, \quad y(\pm\infty) = 0,$$

which is same as

$$\frac{1}{\lambda}y'' = q(x)y, \quad q(x) = \frac{|x| - \lambda}{\lambda}.$$

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Comparing to two-turning point solution, have $x_1 = -\lambda$ and $x_2 = \lambda$. Solvability is

$$\int_{-\lambda}^{\lambda} \sqrt{\lambda - |x|}dx = \frac{4}{3}\lambda^{3/2} = (n + \frac{1}{2})\pi, \quad n = 0, 1, 2, \ldots$$

so that

$$\lambda \sim \left[\frac{3\pi}{4}(n + \frac{1}{2})\right]^{2/3}.$$

Typical eigenfunction: