Asymptotic Methods

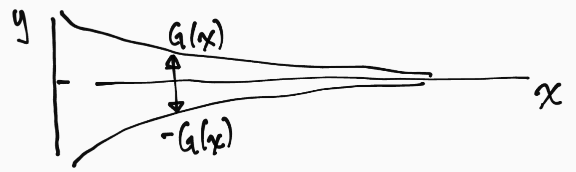
WKB methods: Wave propagation and ray methods

Wave propagation through a slender domain

Consider

$$\epsilon^2 u_{xx} + u_{yy} = u_{tt}, \quad 0 < x < \infty, \quad -G(x) < y < G(x).$$

with $u(x, \pm G(x), t) = 0$ and $u(0, y, t) = f(y)\cos(\omega t)$.



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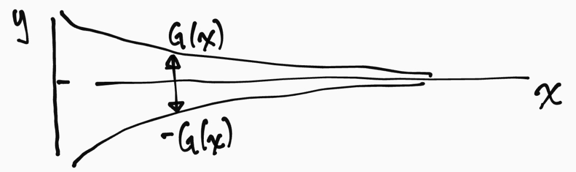

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WKB approximation

$$u \sim \exp(i(\omega t - \theta(x))/\epsilon)[u_0(x, y) + \epsilon u_1(x, y) + \ldots]$$

leads to

$$-\theta_x^2 - i\epsilon(\theta_{xx} u_0 + 2\theta_x u_{0x} + \ldots) + u_{0yy} + \epsilon u_{1yy} + \ldots = -\omega^2(u_0 + \epsilon u_1 + \ldots)$$

Wave propagation through a slender body, cont.

Leading order problem is

$$u_{0yy} + (\omega^2 - \theta_x^2)u_0 = 0, \quad u_0(x, \pm G(x)) = 0.$$

This is an eigenvalue problem (in the y variable), whose solutions are

$$u_0(x, y) = A(x)\sin[\lambda(y + G)]$$

where by boundary condition, $2\lambda G = n\pi$, $n = 1, 2, 3, \ldots$. Therefore

$$\theta_x = \pm\sqrt{\omega^2 - \lambda_n^2}.$$

Note higher modes $\lambda_n > \omega$ decay rather than propagate.

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Next order is

$$u_{1yy} + \lambda_n^2 u_1 = i(\theta_{xx}u_0 + 2\theta_x u_{0x}), \quad u_1(x, \pm G(x), t) = 0.$$

Fredholm solvability implies

$$0 = \int_{-G(x)}^{G(x)} u_0(\theta_{xx}u_0 + 2\theta_x u_{0x})dy = \frac{d}{dx}\int_{-G(x)}^{G(x)} (\theta_x u_0^2)dy,$$

so that $\theta_x \int_{-G}^{G} u_0^2 dy = a$ and $A(x) = a/\sqrt{\theta_x G(x)}$, and solution is superposition of

$$u \sim \frac{a}{\sqrt{\theta_x G(x)}}\exp(i[\omega t \pm \theta(x)/\epsilon])\sin[\lambda_n(y + G)].$$

Wave propagation through a slender body, turning point problem

Turning points x_t are where $2\omega G(x_t) = n\pi$. If $G' < 0$ with $G(\infty) = 0$, then either
(1) exists unique turning point $x_t > 0$, or
(2) the wave decays rather than propagates. This will be true for low frequencies ω.

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With $X = (x - x_t)/\epsilon^{2/3}$, transition solution is

$$u \sim a\text{Ai}(\kappa^{1/3}X) + b\text{Bi}(\kappa^{1/3}X), \quad \kappa = -2\omega^2 (G'/G)|_{x=x_t}.$$

If solution to left of turning point is

$$u \sim \frac{a}{\sqrt{\theta_x G(x)}} \left[a_L \exp(i[\omega t + \theta(x)/\epsilon] + b_L \exp(i[\omega t - \theta(x)/\epsilon])]\right) \sin[\lambda_n(y + G)].$$

and to right is

$$u \sim \frac{a_R}{\sqrt{|\theta_x| G(x)}} \exp(i\omega t - \theta(x)/\epsilon]) \sin[\lambda_n(y + G)],$$

connection formulas imply $a_R = e^{-i\pi/4} a_L$, $b_L = -ia_L$

Rays in wave propagation

Consider the three dimensional wave equation

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For large frequencies $\omega \gg 1$, try WKB expansion

$$v = e^{i\omega\theta(x)}[v_0(x) + \frac{1}{\omega}v_1 + \ldots],$$

leading to

$$|\nabla\theta|^2 = \mu^2(x), \quad \text{(Eikonal)},$$

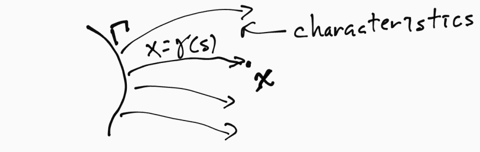
$$2\nabla\theta \cdot \nabla v_0 + (\Delta\theta) \cdot v_0 = 0, \quad \text{(Transport equation)}$$

The eikonal equation

The equation $|\nabla\theta|^2 = \mu^2(x)$ is nonlinear and hyperbolic, and is generally solved with numerical techniques. The solution has a natural geometric characterization, however. Suppose $\theta = 0$ on some surface Γ. Then

$$\theta(x) = \mathrm{dist}(x, \Gamma), \quad \mathrm{dist}(x, x') = \min_{\gamma} \int_0^1 \mu(\gamma(s))\gamma'(s)ds$$

where $\gamma(s)$ is a parameterized curve so that $\gamma(0) = x$ and $\gamma(1) = x'$.



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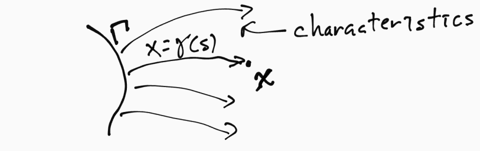


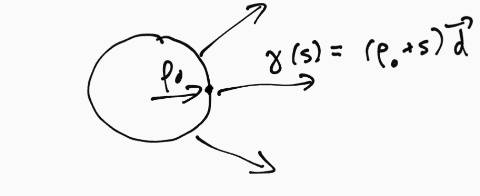
- The integral is simply the time along the curve γ. The minimizing property is Fermat's principle.
- The minimizing curves are characteristics, so that $\gamma'(s) = \nabla\theta(s)$.
- The transport equation is solved along characteristics,

$$\frac{dv_0}{ds}(\gamma(s)) = \nabla\theta \cdot \nabla v_0 = -\frac{1}{2}\Delta\theta(\gamma(s))v_0.$$

Thus if characteristics spread out, $\nabla \cdot (\gamma') = \Delta\theta > 0$ and amplitude shrinks to conserve total power.

Example

Take $\mu = 1$, and suppose $u = 1$ on spherical surface with radius $\rho = \rho_0$. Characteristics are straight lines $x = \gamma(s) = (s + \rho_0)d$ where d is unit direction vector of x.



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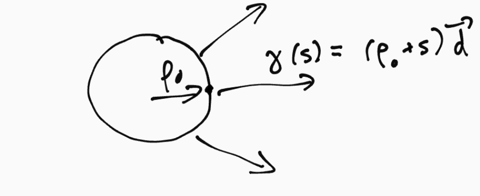

One has $\theta = 0$ and $v_0 = 1$ when $\rho = \rho_0$ or $s = 0$. The radial Laplacian is $\Delta\theta = 2/\rho = 2/(s + \rho_0)$, and therefore transport along characteristic satisfies

$$\frac{dv_0}{ds}(\gamma(s)) = -\frac{1}{2}\Delta\theta(\gamma(s))v_0 = -\frac{v_0}{s + \rho_0}.$$

This equation integrates to $v_0 = \rho_0/\rho$, i.e. intensity drops off as the square of distance.