Homework 1 solutions  
Math 587

1. Find an asymptotic expansion with two non-zero terms for solutions of $\cos x = x/\epsilon$.  
Check your approximation for $\epsilon = 0.3$ against a numerical solution (obtained by, for example, plotting $\cos$ against $-x/\epsilon$).
   
   Expand $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$, leading to $x_0 = 0$, $x_1 = 1$, $x_2 = 0$ and $x_3 = -x_1^2/2$, so that $x \sim \epsilon - \epsilon^3/2$.

2. Find a three term asymptotic expansion of $\ln(1 + e^{\epsilon^{-1}})$ for $\epsilon \to 0$. Show that it is an asymptotic series by using the definition.
   
   Write as 
   \[
   \ln(1 + e^{\epsilon^{-1}}) = \ln(e^{1/\epsilon}(e^{-1/\epsilon} + 1)) = \frac{1}{\epsilon} + \ln(1 + e^{-1/\epsilon})
   \]
   
   If $\epsilon << 1$ then $e^{-1/\epsilon}$ is very small so can Taylor expand $\ln(1 + x)$ about 0,
   
   \[
   \ln(1 + e^{1/\epsilon}) = \frac{1}{\epsilon} + \ln(1 + e^{-1/\epsilon}) = \frac{1}{\epsilon} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n/\epsilon}
   \]

3. Consider $\epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0$ for small $\epsilon$. Investigate all cases of dominant balance, and determine two-term expansions for solutions when appropriate.
   
   Supposing large root scales to leading order like $\epsilon^{-\alpha}$ with $\alpha > 0$ if we use the approximation $x = x_0 \epsilon^{-\alpha}$ in the polynomial have 
   
   $x_1^8 \epsilon^{-8\alpha+1} - x_1^6 \epsilon^{-6\alpha+2} + x_1 \epsilon^{-\alpha} - 2 = 0$
   
   Now want to check for the possibility of dominant balance between various terms

   Case 1: First and Second Terms
   Then $-8\alpha + 1 = -6\alpha + 2$ thus $\alpha = -1/2$. But want $\alpha > 0$ so roots are large. Thus, cannot get dominant balance in this case.

   Case 2: First and Third Terms
   Then $-8\alpha + 1 = -\alpha$ so $\alpha = 1/7$. In this case the first and third terms are $O(\epsilon^{-1/7})$ and the other terms are $O(1)$ thus can have dominant balance between the first and third terms.
Case 3: First and Last Terms
Here \( \alpha = \frac{1}{8} \) but in this case the first and last terms are \( O(1) \) but the third term is not \( O(\epsilon^{-1/8}) \) so don’t have dominant balance between first and last terms.

Case 4: Second and Third Terms
In this case need \( \alpha = \frac{2}{5} \) but then first terms is \( O(\epsilon^{-11/5}) \) while the second and third terms are \( O(\epsilon^{-2/5}) \).

Case 5: Second to Last Terms
Here \( \alpha = \frac{1}{3} \) but while th second and last terms are \( O(1) \) the first term is now \( O(\epsilon^{-5/3}) \) so no dominant balance here.

Case 6: Third and Last Terms
This case requires \( \alpha = 0 \), gives dominant balance.

There are two possible roots, \( x \sim 2 \) and \( x \sim -1/\epsilon^{1/7} \).

4. Find an expansion for the large roots of \( x \tan(x) = 1 \) (hint: it is useful to write this as \( x = \tan^{-1}(1/x) + k\pi \) where \( k \) is large and fixed)

Let \( x = k\pi + x_1 \) where \( x_1 \ll k \), leading to \( x_1 = 1/(k\pi) \). Continuing, let \( x = k\pi + 1/(k\pi) + x_2 \) where \( x_2 \ll 1/k \), leads to \( x_2 = 4/(3n^2\pi^3) \).

5. Let \( A, B \) be nonsingular \( n \times n \) matrices. Find a three-term expansion of \( (A + \epsilon B)^{-1} \) for \( \epsilon \to 0 \).

Suppose \( C \) is of the form \( C = X_0 + \epsilon X_1 + O(\epsilon^2) \) where \( X_i \) are \( n \times n \) matrices.

Substituting this into \( C(A + \epsilon B) = I \) and matching powers of \( \epsilon \) have

\[
\begin{align*}
I &= C(A + \epsilon B) \\
&= (X_0 + \epsilon X_1 + O(\epsilon^2))(A + \epsilon B) \\
&= X_0 A + \epsilon(X_1 A + X_0 B) + O(\epsilon^2)
\end{align*}
\]

So have \( X_0 A = I \) and \( X_1 A + X_0 B = 0 \) thus

\[
\begin{align*}
X_0 &= A^{-1} \quad (O(1)) \\
X_1 A &= -X_0 B \quad (O(\epsilon)) \\
X_1 &= -X_0 BA^{-1} \\
X_1 &= -A^{-1}BA^{-1}
\end{align*}
\]

And so \( C = (A + \epsilon B)^{-1} = A^{-1} - \epsilon A^{-1} BA^{-1} + O(\epsilon^2) \)

6. Find a two-term expansion for \( O(1) \) eigenvalues of the Sturm-Liouville problem

\[
u''(x) + \lambda u = 0, \quad u(0) = 0, \quad u(L) = \epsilon u'(L).
\]
Expanding $u = u_0 = \epsilon u_1 + \ldots$ and $\lambda = \lambda_0 + \epsilon \lambda_1 + \ldots$, $u_0 = \sin(n\pi x/L)$ with a particular eigenvalue $\lambda = (n\pi/L)^2$. The correction solves

$$u_1'' + \lambda_0 u_1 = -\lambda_1 u_0, \quad u_1(0) = 0, \quad u_1(L) = \epsilon u_0'(L).$$

A solvability condition is obtained by multiplying by eigenfunction $u_0$ and integrating by parts:

$$u_1' u_0 - u_1 u_0'|_0^L = -\lambda_1 \int_0^L u_0^2 dx,$$

from which

$$\lambda_1 = \frac{(n\pi/L)^2}{\int_0^L \sin^2(n\pi x/L) dx}.$$