Mochizuki's Corollary 3.12 and my quest for its proof

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> the essence of mathematics lies precisely in its freedom.
>
> Georg Cantor [Ewald, 2005]

This paper summarizes the key ideas of my theory from my papers [Joshi, 2021a, 2023a, 2022a] which provide, with proofs, a clear approach to understanding the claims of [Mochizuki, 2021a,b,c,d]. It includes a discussion of the role of classical Teichmuller Theory in understanding Mochizuki's Corollary 3.12 and also in understanding the proof of geometric Szpiro inequality (§ 1.10–§ 1.21).

This paper addresses Mochizuki's and Scholze's comments on my work (in § 1.6); a discussion of [Mochizuki, 2022] (in § 1.7) and also provides additional mathematical evidence against the widely believed objections of [Scholze and Stix, 2018] regarding [Mochizuki, 2021a,b,c,d] (in § 1.8).

Conclusion is in § 1.25.

§ 1.1 Since the appearance of [Mochizuki, 2021a,b,c,d] papers online in 2012, Mochizuki's work has generated a significant amount of excitement, debates and controversies. One must recognize that there are many reasons for this. A finely tuned presentation of the key ideas of the proof could have averted some of the controversies. However since the appearance of [Scholze and Stix, 2018] there are now two factions, with one believing the validity of his work while the other denying it. The publication of [Mochizuki, 2021a,b,c,d] has not abated this debate. The key contentious issue is the validity of [Mochizuki, 2021c, Corollary 3.12]. The two factions have mostly argued without letting the mathematics speak for itself with abundant clarity. For my discussion of mathematical facts in this context see [Joshi, 2022a, 2021b].

Let me remind the readers that a mathematical assertion may not be falsified by any mathematical diktats but only by a mathematical counter example. For the record, to date, no mathematical counter example to the said corollary has been published.

§ 1.2 My position on [Mochizuki, 2021a,b,c,d], from the time I got interested in the Spring of 2018, has been that I can make a better assessment of its claims provided that I understand exactly what its claims are and the precise mechanism by which the claims may be proved. I believe that now I do have such an insight into his work and I have documented my ideas with proofs in [Joshi, 2019, 2020a,b, 2021a, 2023a, 2021b, 2023b]. Let me add that [Joshi, 2023b] contains a 'Rosetta Stone' for translating between my theory and [Mochizuki, 2021a,b,c].

§ 1.3 The central difficulty with [Mochizuki, 2021a,b,c] and [Mochizuki, 2021c, Corollary 3.12] specifically is how to make sense of the four central assertions it rests on:

(1) The existence of distinct arithmetic holomorphic structures. This breaks down into two related but also separate portions:

(a) the existence of distinctly labeled isomorphs of $(\Pi \twoheadrightarrow G)$ where Π is the tempered fundamental group of a fixed hyperbolic curve over a fixed p-adic field E, $G = G_E$ is the absolute Galois group of E, and $\Pi \twoheadrightarrow G$ is a continuous surjection of topological groups. In [Mochizuki, 2021a,b,c] parlance, each isomorph or instance of this pair is called an *étale-like picture*.

(b) The second portion is subtler: it consists of the value group of an algebraic closure of the p-adic field fixed in part (a) i.e. the value group $\left|\bar{\mathbb{Q}}_p\right|$. The theory requires distinct instances of such value groups (even if an instance of $(\Pi \twoheadrightarrow G)$ is held fixed), with some identifications which allows us to compare valuations in some uniform way. In [Mochizuki, 2021a,b,c] each instance of this monoid datum is called a *Frobenius-like picture.*

(2) The existence of Theta-links (specifically Θ_{gau}-links).

(3) The existence of $\log$-links and $\mathfrak{log}$-links.

(4) The existence of a suitable group whose action moves around the data in (1), (2) and (3).

§ 1.4 To put it succinctly [Mochizuki, 2021c, Corollary 3.12] is an 'averaging' procedure for functions over the space of data (1), (2) and (3). For a given elliptic curve over a number field, spaces of such data exist for all the primes of the number field though one works with the finite set of odd primes of split multiplicative reduction. Here are two examples of functions of arithmetic interest on the space of data (1), (2), (3): (a) the Tate parameter of an elliptic curve over a p-adic field, and (b) the values of theta function at a chosen collection of torsion points (see [Joshi, 2023a] for precise assertions). Because of the existence of the group action (4) one may start with one data point(s) exhibiting (1), (2) and (3) and consider its orbit under group in (4) and carry out the averaging procedure over this orbit. Roughly speaking, this is essentially the strategy of [Mochizuki, 2021c]. *So it is crucial that the existence of such data is convincingly established in the first place.*

§ 1.5 My work [Joshi, 2021a, 2022a, 2023a] *canonically* establishes the existence of data **(1)**, **(2)**, **(3)** and **(4)**. This is done by a precise definition of arithmetic holomorphic structures and the precise demonstration of why they may be so anointed. To be sure, Mochizuki coined this phrase and his definition of arithmetic holomorphic structure is different from mine, but I found his definition difficult to work with in clearly establishing the existence of the above data. On the other hand, my definition of arithmetic holomorphic structure provides arithmetic holomorphic structures in Mochizuki's sense [Joshi, 2022a].

The arithmetic Teichmuller space I construct is essentially the space of all arithmetic holomorphic structures one may put on $X/\mathbb{Q}_p$!

The precise notion of arithmetic holomorphic structure detailed in [Joshi, 2022a] is quite closely related to Scholze's Theory of Diamonds [Scholze, 2017]. To put my work [Joshi, 2021a, 2022a] in *the most simplistic terms*, each arithmetic holomorphic structure provides a triple $(X_E^{an}, X_K^{an}, X_K^{an} \to X_E^{an})$ consisting of Berkovich analytic spaces X_E^{an} and X_K^{an} and a morphism of analytic spaces $X_K^{an} \to X_E^{an}$ and an untilt, $(E \hookrightarrow K, K^\flat \simeq F)$, of some perfectoid field F of characteristic $p > 0$ (as I observe in my papers, the theory must work with all F even though calculations for the Diophantine inequalities can be performed by taking $F = \mathbb{C}_p^\flat$), and this allows me to establish **(1)** above. Notably I demonstrate in [Joshi, 2021a, 2022a] that each arithmetic holomorphic structure provides a pair $(E \hookrightarrow K)$ which provides an instance of Mochizuki's étale-like picture as the morphism of tempered fundamental groups of the $\pi_1(X_K^{an}) \hookrightarrow \pi_1(X_E^{an}) \twoheadrightarrow G_E$ (with $(\Pi = \pi_1(X_E^{an}) \twoheadrightarrow G_E)$ providing **(1a)** above). In [Joshi, 2023a] I observe that tilt data $(K^\flat \simeq F)$ provides the value group $|K^\flat|$ and the isomorphism $K^\flat \simeq F$ provides a way of comparing valuations (of elements of K) in the fixed value group $|F|$ (i.e. **(1b)** above). Moreover, tilting i.e. construction of $K^\flat$ from K requires working with p^{th}-powers (i.e. Frobenius lifts and its powers) and hence an arithmetic holomorphic structure provides, via the $(K^\flat \simeq F)$ portion of the datum, a Frobenius-like picture in the sense of [Mochizuki, 2021a,b,c]. In particular one has many instances of arithmetically inequivalent Frobenius-like pictures (an assertion which is needed in [Mochizuki, 2021c]). Existence of $\mathfrak{log}$-links and Theta-Links (i.e **(2)** and **(3)**) is detailed in [Joshi, 2023a]. As for **(4)** the relevant group and its action is explicated in detail in [Joshi, 2021a, 2022a] and explicitly allows me to demonstrate how arithmetic holomorphic structures move under this action (a similar assertion has been made by Mochizuki in [Mochizuki, 2021c, Theorem 3.11]–my version is the precise version of Mochizuki's assertion).

§ 1.6 *Let me say that I am a deep admirer of the works of Shinichi Mochizuki and Peter Scholze.* But the truth is neither Mochizuki nor Scholze have anticipated that modern p-adic Hodge Theory as developed by many (including Scholze) has significant ramifications for [Mochizuki, 2021a,b,c] (as is detailed in my papers [Joshi, 2021a], [Joshi, 2023a], [Joshi, 2021b], [Joshi, 2023b]). My original work is currently the best mathematical (and conceptually cleaner) approach for proving Mochizuki's Corollary 3.12 (based on principles laid out by Mochizuki). In my theory, Mochizki's work on the abc-conjecture and the theory of perfectoid fields (due to Scholze and others) are inextricably intertwined in establishing convincing proofs of the claims of [Mochizuki, 2021a,b,c].

In [Mochizuki, 2022], Mochizuki has alluded to my work as a "fabricated version." Frankly, I am puzzled by Mochizuki's response to my work because he pioneered the use of classical p-adic Hodge Theory in Anabelian Geometry with remarkable success. In [Joshi, 2019], I demonstrated how one can algebraize Mochizuki's idea of keeping multiplicative structures (i.e. multiplicative monoids) of a ring fixed and deform the ring structure. As a consequence of [Joshi, 2019] one sees that the theory of perfectoid fields [Scholze, 2012], [Fargues and Fontaine, 2018] also rest on similar deformations (as is detailed in my papers [Joshi, 2021a], [Joshi, 2023a], [Joshi, 2021b], [Joshi, 2023b]). [Notably in [Fargues and Fontaine, 2018] and [Scholze, 2012] the multiplicative monoid $\widehat{\mathbb{G}}_m(\mathscr{O}_F)$ remains fixed.]

My work follows all the rules laid out in [Mochizuki, 2021a,b,c], but arrives at the central ideas of [Mochizuki, 2021a,b,c] via [Joshi, 2019], [Joshi, 2020a], [Joshi, 2021a, 2022a]. My work replaces Mochizuki's awkward formalism completely and provides a clear way of establishing most of the important claims of [Mochizuki, 2021a,b,c]. This has allowed me to break the impasse which had been reached in the understanding of Mochizuki's claims with the appearance of [Scholze and Stix, 2018].

Scholze has recently said that my work is "linguistic trickery" because of my use of algebraically closed perfectoid fields and has asserted that perfectoid fields have nothing to do with anabelian geometry. Mochizuki (in 2020) and Mochizuki and Y. Hoshi (in 2021) have also made an identical assertion. *Let me say clearly that this point has been explicitly falsified in [Joshi, 2022a, 2021a].* As is demonstrated there, algebraically closed perfectoid fields enter anabelian geometry–more precisely–the theory of tempered fundamental groups (and hence [Mochizuki, 2021a,b,c,d]) through the role of arbitrary geometric basepoints. Mochizuki clearly states that unrelated or arbitrary geometric basepoints are required in his theory (see

[Mochizuki, 2021a, § I3]). So this denial of the role of perfectoid fields in the context of [Mochizuki, 2021a,b,c] is quite incorrect.

Once this is recognized, assertions such as [Scholze and Stix, 2018, Remark 9] also fall apart as they require identification of fundamental groups arising from all basepoints. Let me say why it is important to discuss this issue. Since the appearance of [Scholze and Stix, 2018] in 2018, [Scholze and Stix, 2018, Remark 9] has been at the foundation for the widely believed assertion (of Scholze-Stix) that Mochizuki's strategy is fundamentally unworkable. My work demonstrates that this is simply not the case.

Just as one may identify fundamental groups of Riemann surfaces of fixed topological type at the expense of ignoring their complex analytic structures, my work shows that tempered fundamental groups may be identified at the expense of Berkovich analytic structures (more precisely arithmetic holomorphic structures). Let it be clear that in [Mochizuki, 2021a,b,c] as well as in my works, one wants to work with arithmetic holomorphic structures and not with fundamental groups per se. My discovery (in [Joshi, 2021a, 2022a, 2023a]) is that certain operation and symmetries permitted by modern p-adic Hodge Theory allow us to change Berkovich analytic structures (more precisely change arithmetic holomorphic structures) and this drives the theory of tempered fundamental groups in the context of [Mochizuki, 2021a,b,c]. Readers may want to recall that Mochizuki's famous works on anabelian geometry, such as [Mochizuki, 1996, 1999], are driven by classical p-adic Hodge Theory.

§ 1.7 Let me address [Mochizuki, 2022]. I will limit this discussion to its mathematical claims. My *personal* reading of [Mochizuki, 2022] is that it tacitly recognizes that the problems many have had with [Mochizuki, 2021a,b,c,d] are real, and it goes to suggest a solution to these problems. It asserts that these difficulties can be made to disappear if readers simply realign their logic to a different set of rules and then re-read [Mochizuki, 2021a,b,c,d] with this new set of rules (unfortunately no discussion of this logic appears in [Mochizuki, 2021a,b,c,d]). One conclusion which *may* be drawn from [Mochizuki, 2022] is that the issues in [Mochizuki, 2021a,b,c,d] require an external circumvention which is (now) provided by [Mochizuki, 2022].

Let me address Mochizuki's discussion of one such suggestion provided in [Mochizuki, 2022, Example 3.2.2, page 77]. Some readers may find the rest of this paragraph (i.e. § 1.7) a bit technical and may wish to skip over to § 1.8. Following its notation let me write $k = \mathbb{Q}_p$ (or some p-adic field) and let $\bar{k}$ be an algebraic closure of k, let $q \in k$ with $|q|_k < 1$. For a natural number n, let

$$\mathscr{F}_n = \mathscr{O}^*_{\bar{k}} \times (q^n)^{\mathbb{Z}} \subset \bar{k}^*$$

(considered as multiplicative monoids).

Then Mochizuki wants to consider the isomorphism of monoids

$$\mathscr{F}_n \xrightarrow{\simeq} \mathscr{F}_1 \tag{1.7.1}$$

and asserts that in [Mochizuki, 2021a,b,c] one wants to glue data along such isomorphisms.

First of all such isomorphisms of monoids obviously exist. The issue in [Mochizuki, 2021a,b,c] is not about the existence of such isomorphisms but whether or not such an isomorphism of monoids arises from some arithmetic geometric data associated to a given hyperbolic curve X/k of topological type $(1, 1)$ with q being the Tate parameter of X/k.

Using the techniques of [Joshi, 2023a], let me sketch a proof of the fact established in [Joshi, 2023b] that a precise version of this gluing of monoids does indeed arise from distinct arithmetic geometric data ([Joshi, 2023b]) and that it arises from incompatible ring/field structures as is required in [Mochizuki, 2021a,b,c,d] according to [Mochizuki, 2022, Example 3.2.2, page 77]. [A detailed and general version of my result and its proof appears in [Joshi, 2023b].]

Using methods I have developed in [Joshi, 2022a, § 6] one can prove that one may find, for each $n = 1, 2, 3, \ldots$, an arithmetic holomorphic structure providing embeddings $k \hookrightarrow K_n$ into algebraically closed perfectoid fields K_n such that $|p|_{K_n} = |p|^n_{K_1}$. Now let q_n be the Tate parameter of Tate curve $X^{an}_{K_n}$. Then for each $n = 1, 2, 3, \ldots$ one has an isomorph $\mathscr{F}_1(k \hookrightarrow K_n)$ of Mochizuki's monoid $\mathscr{F}_1$ given by computing this monoid in the datum $k \hookrightarrow K_n$. One has

$$\mathscr{F}_1(k \hookrightarrow K_n) = \mathscr{O}^*_{\bar{k}} \times q_n^{\mathbb{Z}}$$

here $k \hookrightarrow K_n$ provides a copy of the algebraic closure $\bar{k} \subset K_n$ of k (and so readers must be cautious in reading the statements below–each side of the equality refers to its own copy of $\bar{k}$). So for each $n = 1, 2, 3, \ldots$ one has an abstract isomorphism of monoids

$$\mathscr{F}_1(k \hookrightarrow K_n) = \mathscr{O}^*_{\bar{k}} \times q_n^{\mathbb{Z}} \xrightarrow{q_n \longmapsto q_1} \mathscr{O}^*_{\bar{k}} \times q_1^{\mathbb{Z}} = \mathscr{F}_1(k \hookrightarrow K_1), \tag{1.7.2}$$

(because both refer to the same object computed for two different arithmetic holomorphic structures) and also the equality of valuations

$$|q_n|_{K_n} = |q_1|^n_{K_1},$$

and typically each $k \hookrightarrow K_n$ arises from a distinct arithmetic holomorphic structure (by construction). At any rate, the isomorphism (1.7.2) is naturally provided by my approach and is the precise version of what is needed in [Mochizuki, 2021a,b,c,d].

This isomorphism can be recast as an isomorphism $\mathscr{F}_n \simeq \mathscr{F}_1$ in Mochizuki's style (1.7.1) as follows. Since the unit groups appearing above are divisible and one may identify $\bar{k} \subset K_n$ abstractly with $\bar{k} \subset K_1$ and view q_n, q_1 in one common field then one may write the monoids appearing in the above equation as

$$\mathscr{O}^*_{\bar{k}} \times q_n^{\mathbb{Z}} \simeq \mathscr{O}^*_{\bar{k}} \times (q_1^n)^{\mathbb{Z}}.$$

This is the shape of the isomorphism asserted in [Mochizuki, 2022, Example 3.2.2].

However, it is not the mere gluing of monoids that is at issue in [Mochizuki, 2021a,b,c], but the existence of the arithmetic holomorphic structures (for Mochizuki this is a surjection $\Pi \to G_k$) which provide such a gluing. Notably my clear understanding of the theory is that one wants (1.7.2) and not the simplistic version (1.7.1) for the theory to work.

Mochizuki's assertion in [Mochizuki, 2022] is that one must consider $\mathscr{F}_n$ **and** $\mathscr{F}_1$ for his theory instead of $\mathscr{F}_n$ **or** $\mathscr{F}_1$ ([Mochizuki, 2022, Example 2.4.8, page 58]). My observation is that this is a correct assertion but not for the reasons Mochizuki provides (namely declaring that the logic of the theory is different), but because $\mathscr{F}_n$ is an avatar of $\mathscr{F}_1$ for a verifiably distinct arithmetic holomorphic structure! [This is exactly what happens in classical Teichmuller Theory which provides many (metrically) distinct avatars of a given Riemann surface i.e. Teichmuller theory provides many distinct Riemann surfaces of a given, fixed moduli.]

The question for me has never been whether or not the copies of the monoids $\mathscr{F}_1, \mathscr{F}_n$ should be considered redundant, but how does one distinguish them as arising from distinct arithmetic and geometric data. This is also my reading of the relevant portion of the objections of [Scholze and Stix, 2018, Scholze, 2021]. My theory answers this question, *at least for me,* and my solution (1.7.2) can also be applied to [Mochizuki, 2021a,b,c,d]. Hence I am not persuaded that [Mochizuki, 2022] adequately addresses the core questions related to [Mochizuki, 2021a,b,c,d]. On the other hand, many of Mochizuki's claims in [Mochizuki, 2021a,b,c] and [Mochizuki, 2022] do have clear proofs in my theory.

§ 1.8 My work has made foundational inroads in arriving at an understanding of Mochizuki's claims and dismantling many of the arguments against his work–the principal ones being [Scholze and Stix, 2018, Scholze, 2021].

Note that Scholze and Stix, especially [Scholze and Stix, 2018, Remark 9], cite [Mochizuki, 2015, Theorem 1.9 and Corollary 1.10] as evidence against [Mochizuki, 2021c, Corollary 3.12] (this argument is repeated by Scholze on Peter Woit's Blog [Scholze et al., April 2020, Page 1] and on MathOverFlow).

My fundamental observation is that Scholze and Stix have failed to distinguish that [Mochizuki, 2015] deals with moduli theory (i.e. determination of the isomorphism class of the curve) while [Mochizuki, 2021a,b,c,d] uses Teichmuller Theory (as should be clear from the very title of Mochizuki's [Mochizuki, 2021a,b,c,d]). There is no mention of Teichmuller Theory in [Scholze and Stix, 2018]. Teichmuller Theory is a finer theory than moduli theory and in the classical case it weakens the notion of conformal equivalence of Riemann surfaces and provides a richer geometric theory from which moduli theory emerges as a special case. Any argument that moduli theory exists and therefore Teichmuller Theory cannot exist is an incorrect conclusion.

Once this point is understood, it is clear that [Scholze and Stix, 2018, Remark 9] which implies that [Mochizuki, 2021a,b,c,d] has no valid strategy, and hence has no mathematical merit, is itself a flawed argument. [The core idea of [Mochizuki, 2021a,b,c,d] is to average over a suitable Teichmuller data–my theory constructs a space of such Teichmuller data (i.e. a Teichmuller space) explicitly. Unlike [Mochizuki, 2022], I provide mathematical proofs of the existence of Teichmuller Theory in the arithmetic context.]

I hope that this mathematical understanding that I bring to [Scholze and Stix, 2018] objections will put to rest the Scholze and Stix, Remark 9 argument against [Mochizuki, 2021a,b,c,d] once and for all and its mathematical resolution presented in my work.

§ 1.9 Over the past years, many attempts have been made in deciphering Mochizuki's claims regarding the abc-conjecture. As for me, nothing but clear mathematics will suffice. The abc-conjecture is ultimately about remarkable properties of Numbers as we teach in our Mathematics Curricula and so its proof must be clearly laid out. I have chosen to present my work with this *optik* in mind. If my work appears elementary to *some readers*, it is entirely because of my desire to make my work accessible to all mathematicians (and so experts on both sides of the debate should have no difficulty in understanding the proofs in my papers). *Readers should not conflate clarity of my work with triviality.*

§ 1.10 Here is a more accessible description of Arithmetic Teichmuller Theory developed in my papers and how it relates to Mochizuki's work and some insight into [Mochizuki, 2021c, Corollary 3.12] from the point of view of classical Teichmuller Theory. This will require some familiarity with the theory of Riemann surfaces and with classical Teichmuller Theory (some standard texts are [Imayoshi and Taniguchi, 1992], [Nag, 1988], [Lehto, 1987]). When I discuss the arithmetic case, I will suppose that X is a geometrically connected, smooth quasi-projective variety over a p-adic field which I will take to be $\mathbb{Q}_p$ for simplicity. In Mochizuki's context X is additionally required to be a hyperbolic curve.

§ 1.11 Suppose Σ is a fixed closed Riemann surface of genus $g \geq 1$. Suppose $\mathscr{T}_\Sigma$ is the Teichmuller space of Σ. Let $\Sigma' \in \mathscr{T}_\Sigma$ be another Riemann surface in the Teichmuller space $\mathscr{T}_\Sigma$ of Σ. This means one has a quasi-conformal mapping $f : \Sigma \to \Sigma'$ which is in fact a homeomorphism of the underlying topological spaces $|\Sigma'| \xrightarrow{\simeq} |\Sigma|$. In Teichmuller Theory one usually fixes the underlying topological space $|\Sigma|$ and works with many different complex structures (or quasi-conformal data) on $|\Sigma|$ corresponding to points of $\mathscr{T}_\Sigma$.

Suppose that γ is a simple closed curve on Σ and suppose that one is interested in the length of γ (or more generally some function of the length of γ) for the metric given by the holomorphic structure of Σ. If $f : \Sigma \to \Sigma'$ is a quasi-conformal mapping, then one may view the curve γ as a curve on Σ' and measure its length for the metric given by Σ'. There is a natural well-defined procedure to do this (for instance see [Imayoshi and Taniguchi, 1992]). This allows us to consider the length of γ in Σ' as Σ' varies over all of $\mathscr{T}_\Sigma$, or Σ' varies over a conveniently chosen subset $S \subset \mathscr{T}_\Sigma$ containing the point (of interest) corresponding to Σ and one can use this set to bound the length of γ on Σ. For example, one trivial bound is to take the supremum over lengths of γ in Σ' as Σ' varies in our chosen subset $S \subset \mathscr{T}_\Sigma$.

§ 1.12 Here is one natural choice of a subset of $\mathscr{T}_\Sigma$ containing the point corresponding to Σ (I will identify Σ with the point corresponding to it in $\mathscr{T}_\Sigma$). Now consider the Teichmuller mapping

$$\mathscr{T}_\Sigma \to \mathscr{M}$$

which maps $\Sigma' \in \mathscr{T}_\Sigma$ to its isomorphism class $[\Sigma'] \in \mathscr{M}$ in the moduli of Riemann surfaces of the same topological type as Σ. This mapping is the canonical quotient mapping of $\mathscr{T}_\Sigma$ by the action of the Teichmuller modular group. Let

$$S_{[\Sigma]} = \{\Sigma' \in \mathscr{T}_\Sigma : [\Sigma'] = [\Sigma] \in \mathscr{M}\} \subset \mathscr{T}_\Sigma$$

be the fiber of the Teichmuller mapping over the isomorphism class $[\Sigma] \in \mathscr{M}$ of Σ.

The set $S_{[\Sigma]}$ can be identified with the orbit of Σ under the action of the Teichmuller modular group. The metric provided by each point $\Sigma' \in S_{[\Sigma]}$ varies but the isomorphism class of every point $\Sigma' \in S_{[\Sigma]}$ coincides with the isomorphism class of Σ in the moduli $\mathscr{M}$.

Notably even if the isomorphism class of Σ is held fixed, metrically one sees some non-trivial variation among the $\Sigma' \in S$ and it is possible to bound length of γ using metrics provided by the Riemann surfaces corresponding to points $\Sigma' \in S_{[\Sigma]}$. More sophisticated versions of this argument in fact do occur in the literature on Teichmuller and moduli spaces of Riemann surfaces (for example see a survey of M. Mirzakhani's work in [Wright, 2019]).

Broadly speaking, Mochizuki's Corollary 3.12 is, at least *in spirit*, this sort of method of bounding a metrically dependent quantity in the presence of a non-trivial variation of (p-adic) metrics. Notably Mochizuki's log-volume computation in Corollary 3.12 should be understood as being similar *in spirit* to the computation of the volume of $\mathscr{M}_{1,1}$ via MacShane's Identity due to M. Mirzakhani (see [Wright, 2019]). Notably, Mirzakhani's computation of the volume $\mathscr{M}_{1,1}$ requires working with suitably extended moduli or Teichmuller space $\mathscr{M}^*_{1,1}$ where the computations can be transparently performed. Similarly Mochizuki's theory (and my theory) requires working with a suitably extended moduli (or Teichmuller) spaces. My approach to [Mochizuki, 2021c, Corollary 3.12] is detailed in [Joshi, 2023a]. My observation in [Joshi, 2023a,b] is that once one has many distinct arithmetic holomorphic structures and other relevant results in place, many versions of Mochizuki's Corollary 3.12 can be formulated (even if one is dealing with several primes instead of one prime as [Joshi, 2023a] does and such a formulation appears in [Joshi, 2023b]). Each version depends on the choice, at each prime, of a suitable set for collating theta-values arising from different arithmetic holomorphic structures; Mochizuki's choice of this set at each prime is a suitable Galois cohomology group and my choice (at each prime) is the Fargues-Fontaine ring $B_p = B_{\mathbb{C}_p^\flat,\mathbb{Q}_p}$.

§ 1.13 If one looks at Σ and $\Sigma' \in \mathscr{T}_\Sigma$ through the narrow lens of fundamental groups, one misses the quasi-conformal structures (and varition of metrics) of Σ, Σ' completely. Notably

Teichmuller Theory (classical, Mochizuki's and my theory) are not about fundamental groups per se but about the variation of the (arithmetic geo)metric data i.e. of *arithmetic data*. [Social Media discussions of [Mochizuki, 2021a,b,c,d] make it clear that this point has not been understood at all.]

§ 1.14 In Diophantine Geometry, heights are dependent on p-adic and archimedean metrics and for this purpose one should work with more geometric objects which allow us to deal with metric structures and their variation directly.

This is why I work with rigid analytic spaces in the sense of Berkovich's Theory instead of using the theory of schemes or fundamental groups as Mochizuki does. My analytic approach is similar to the case of Riemann surfaces and quasi-conformal mappings. Analytic spaces, of course, provide Riemann surfaces and p-adic curves (respectively) and also their fundamental groups. This provides full compatibility with Mochizuki's group theoretic approach.

§ 1.15 My work in fact produces the exact analog of the classical picture in the p-adic context. Let $X/\mathbb{Q}_p$ be a geometrically connected, smooth quasi-projective curve over $\mathbb{Q}_p$ (say). I observe that non-trivial deformations of the pair $X/\mathbb{Q}_p$ and K alg. closed perfectoid field containing $\mathbb{Q}_p$ exist in which the isomorphism class of $X/\mathbb{Q}_p$ remains fixed but the $\mathbb{Q}_p$-isomorphism class of the analytic space X^{an}_K (note that $K/\mathbb{Q}_p$ is far from being a finite extension or even a finitely generate extension) moves because the field K itself admits non-trivial p-adic deformations. This is best demonstrated when $(X/\mathbb{Q}_p, X/K)$ are projective or proper and this I carry out in my [Joshi, 2021a]. The pair of analytic spaces $(X^{an}_{\mathbb{Q}_p}, X^{an}_K)$ provide the triple of data

$$\pi_1^{temp}(X^{an}_{\mathbb{Q}_p}) \hookleftarrow \pi_1^{temp}(X^{an}_K)$$

consisting of the tempered fundamental group of $X^{an}_{\mathbb{Q}_p}$ and the geometric tempered fundamental group X^{an}_K and together with the inclusion of the latter in the former (both the groups computed using a common choice of a K-geometric basepoint).

There is no mathematical disagreement with my claim and its proofs in [Joshi, 2021a, 2022b,a]. Several experts on tempered fundamental groups (including Jacob Stix in 2020–[Joshi, 2022b]) have looked at my proofs. But no mathematical issues with my claims have emerged to date. So the proofs are quite firm at this point (some typos may occur of course). Since [Scholze and Stix, 2018], only Scholze has provided further published commentary on Mochizuki's work (through [Scholze, 2021] and public online discussion). I would welcome additional comments from Stix about [Mochizuki, 2021a,b,c,d], whether his position has changed or not.

§ 1.16 Now to address the question of why one should bring in perfectoid fields K in the anabelian context at all? *This is a perfectly valid question.* If truth be told, many anabelian geometers and algebraic geometers may feel that this is an external imposition on anabelian geometry.

However, the answer to this question rests on the following three points:

(1) The minimal geometric data required to define the tempered fundamental group of $X/\mathbb{Q}_p$ consists of

(a) $X/\mathbb{Q}_p$ and

(b) a morphism $\mathscr{M}(K) \to X^{an}_{\mathbb{Q}_p}$ of analytic spaces, where K is an algebraically closed complete, rank one valued field K, containing an isometrically embedded $\mathbb{Q}_p$, to serve as a geometric basepoint. Such algebraically closed complete valued fields K

are (algebraically closed) perfectoid fields. The tempered fundamental group itself is independent of the choice the geometric basepoint and hence of K [André, 2003, Chap III, § 1.4.4].

(2) Moreover if one wants to work with arbitrary isomorphs of tempered fundamental groups, as Mochizuki does, then this in particular means arbitrary algebraically closed perfectoid fields K (as above) are necessarily allowed in [Mochizuki, 2021a,b,c].

(3) *Now for the most important point*: from Mochizuki's discussion (in [Mochizuki, 2021a, § I3, page 25]) of the fundamental role of arbitrary basepoints in [Mochizuki, 2021a,b,c] and the fact the key operations of the theory such as the theta-link and the log-links require distinct basepoints, it becomes clear that [Mochizuki, 2021a,b,c] is a theory which requires working with many distinct basepoints simultaneously. However, fundamental groups arising from distinct basepoints, while being non-canonically isomorphic, may not be identified with each other in [Mochizuki, 2021a,b,c]. So arbitrary perfectoid fields are inevitably present in the theory!

§ 1.17 So my introduction of algebraically closed perfectoid fields in this context is quite a natural one based on requirements of [Mochizuki, 2021a,b,c]. Now my observation is that whether one works with Mochizuki's point of view of working with all the arbitrary isomorphs of the tempered fundamental group (in [Mochizuki, 2021a,b,c,d]) or whether one works with the collection of the data consisting of $(X/\mathbb{Q}_p, X^{an}/K, K$-geometric basepoint of $X^{an}_E)$ as is done in [Joshi, 2022a], [Joshi, 2021a], collections of each of these data have the look and feel of the fiber of the Teichmuller mapping $\mathscr{T}_\Sigma \to \mathscr{M}$ (considered above) because the isomorphism class of $X/\mathbb{Q}_p$ remains fixed in [Mochizuki, 2021a,b,c], [Joshi, 2021a] and [Joshi, 2023a].

§ 1.18 One important point which should be understood is this: as every abstract isomorphism between discretely valued fields is in fact an isomorphism of valued fields, so it is not possible to construct arithmetic Teichmuller Theory by working with discretely valued fields alone! Doing so leads to fallacious conclusions of the sort which have been made in the context of [Mochizuki, 2021a,b,c].

§ 1.19 Importantly the p-adic results I establish have clear classical analogs and one important observation is that Mochizuki's Indeterminacy of Type II (required in the statement of [Mochizuki, 2021c, Theorem 3.11, Corollary 3.12]) has a classical analog. It corresponds to the Virasoro action on Teichmuller and Moduli spaces which has been well-studied in Physics literature as well as algebraic geometry literature ([Beilinson and Schechtman, 1988], [Kontsevich, 1987]). The precise p-adic analog of this action is established in [Joshi, 2021a, 2022a]. This action is an important ingredient in [Mochizuki, 2021c, Theorem 3.11] (and hence in the formulation of [Mochizuki, 2021c, Corollary 3.12]).

Notably I am able to prove that arithmetic holomorphic structures move non-trivially under this action–this is asserted by Mochizuki in [Mochizuki, 2021c, Theorem 3.11 and Corollary 3.12] but this claim has proved rather difficult *for me* to verify using Mochizuki's approach.

But perhaps more importantly, I demonstrate that Mochizuki's Indeterminacies of all types arise naturally from geometry and arithmetic properties of the spaces I construct. More precisely Mochizuki's Indeterminacies (see [Joshi, 2021b] for my discussion of Indeterminacies) arise from the forgetful functor

$$(X^{an}_{\mathbb{Q}_p}, X^{an}_K) \longmapsto \left(\pi_1^{temp}(X^{an}_{\mathbb{Q}_p}) \hookleftarrow \pi_1^{temp}(X^{an}_K)\right)$$

which forgets arithmetic holomorphic structures and remembers only the tempered fundamental group and the geometric tempered fundamental (sub)group. This is similar to the functor from classical Teichmuller Theory

$$\mathscr{T}_{\Sigma} \ni \Sigma' \longmapsto \pi_1(\Sigma')$$

which forgets the complex structure of Σ' and remembers only the fundamental group. Notably, numerical or geometric quantities which are associated to Σ' i.e. to a complex structure on the topological space $|\Sigma|$ necessarily appear indeterminate from the point of view of this functor. So my analogy with classical Teichmuller spaces is to the point, and my theory provides a far better way to understanding and proving Mochizuki's claims.

§ 1.20 One important point which needs to be made here is that for Diophantine applications one needs to compute local arithmetic degrees in a uniform way as the arithmetic holomorphic structure varies. This requires working with arithmetic holomorphic structures ([Joshi, 2022a, 2021a]) arising from a fixed algebraically closed perfectoid field F of characteristic $p > 0$ (for example $\mathbb{C}_p^\flat$). Mochizuki has a similar requirement in [Mochizuki, 2021a,b,c]. His theory tracks value group information separately using Frobenioids and this is central to [Mochizuki, 2021c, Corollary 3.12]. My approach using arithmetic holomorphic structures is detailed in [Joshi, 2022a] and [Joshi, 2021a] and [Joshi, 2023a].

§ 1.21 It is true that Classical Teichmuller Theory has seldom been used in Diophantine geometry in the past. But this does not mean it cannot be used. The proofs of the geometric Szpiro inequality [Amorós, Bogomolov, Katzarkov, and Pantev, 2000], [Zhang, 2001] take place in the presence of classical Teichmuller data of a Riemann surface (Mochizuki's discussion of these two papers appears in [Mochizuki, 2016]). These proofs essentially work with orbits of the mapping class group (see my discussion of this in § 1.12). To me, these proofs already demonstrate the role that classical Teichmuller Theory can play in global Diophantine problems in the geometric case. On the other hand [Scholze and Stix, 2018, Scholze, 2021] offers no discussion of [Amorós et al., 2000], [Zhang, 2001], [Mochizuki, 2016]. My parallel reading of these three works has played an important role in the evolution of my ideas on Arithmetic Teichmuller Spaces in the context of Diophantine applications. History of Diophantine Geometry, in the past hundred and fifty years, shows that an understanding of the geometric case has often paved the road to the proofs in the arithmetic case. This is an important point which should not be forgotten.

§ 1.22 To be sure, my papers detail the local picture i.e. the view at every prime. Mochizuki starts with the global picture which makes his approach difficult to understand. My approach is the opposite. I came to recognize early on in my investigations that the central difficulty which many have had in even imagining that an assertion such as [Mochizuki, 2021c, Corollary 3.12] might be true is because [Mochizuki, 2021a,b,c] offers no convincing demonstration of the existence of distinct arithmetic holomorphic structures and this existence is a local assertion to be established at each prime. So I felt the need to firmly establish the local picture before one can hope to make any global Diophantine assertions–*this is not any limitation of my methods. Global theory is dealt with in [Joshi, 2023b].*

In [Joshi, 2023a] I prove my version of [Mochizuki, 2021c, Corollary 3.12] based on the ideas elaborated in that paper and in [Joshi, 2021a]. My paper [Joshi, 2023a] works with one prime for simplicity, but in general one needs to work with several primes simultaneously and this will appear in [Joshi, 2023b] (proofs of both of my versions is similar). Notably my approach to the general case demonstrates how (and why) the tensor packet structure asserted by Mochizuki in [Mochizuki, 2021c, § 3] arises.

§ 1.23 Recently there was some discussion of my work on Mathoverflow, to which I have posted a brief response on David Roberts's blog. Here, I address the claims posted there about lack of applications of my work to Diophantine inequalities. Diophantine inequalities require bounding Arakelov heights of points. This theory of heights is about metrics both archimedean and non-archimedean (my discussion of the geometric case of Szpiro inequality is in § 1.21). Mochizuki's claim is that greater insight (and better bounds) may be obtained by considering a natural family of metrics (more precisely–arithmetic holomorphic structures) containing the given one and averaging over such a family. That such an averaging is possible (even in the non-archimedean case) is a point I prove with great precision in my papers.

§ 1.24 There is no doubt that I make a rather unconventional usage of mathematics familiar to some mathematicians. But this way of thinking is a completely valid usage of the underlying mathematics and the objects I describe exist, whether one likes it or not, and they have the properties which I have established in my papers.

§ 1.25 My work, building on Mochizuki's work, offers a different way of thinking about Numbers in the context of Diophantine Geometry. I claim that the critical insight which our works offer is that in a precise sense p-adic arithmetic (and consequently p-adic Geometry) is both remarkably rigid and remarkably fluid. For the past hundred years number theorists have mostly worked with p-adic arithmetic in its rigid aspect. The central thesis of our works is that new insights into Arithmetic and Diophantine Geometry may be gained by incorporating the fluid aspect–because one has the freedom to do so. Mochizuki and I have arrived at this viewpoint by entirely different but convergent paths.

§ 1.26 *If there are any mathematical questions or challenges to my work I will be happy to answer them.* I thank David M. Roberts for suggestions which have improved the readability of this manuscript.

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