Some comments on the local/global arguments raised by Mochizuki and Scholze

Kirti Joshi

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0.1

I address Mochizuki's and Scholze's comments about local/global issues in my proof of Theorem 9.11.1 in [Constr. III] and its relationship with the proof of the main theorems of [Const. IV]. Note that [Constr. IV] closely follows [IUT 4]. This revised document was also emailed to Mochizuki and also to Scholze.

0.2

Two important point of clarification:

(1) There is a global action of L^* on the space of arithmeticoids which is described in [Constr. II(1/2), Theorem 4.2.3, Theorem 4.4.1, Corollary 4.4.2] (similar properties, especially Cor. 4.4.2, has been asserted by Mochizuki in [IUT3] as being part of global aspects of his theory). I have been a bit lax in explicating the role of this action in current version of [Constr. III]. But this will be done in the next update of [Constr. III]. The role of this global action of L^* in the theory of heights in the context of the abc-conjecture and Diophantine Geometry is already indicated in the examples discussed in [Constr. II(1/2), §8.5].

(2) The recent update of [Constr. II(1/2)] includes two innovations [Constr. II(1/2), Thm 5.9.1, Remark 5.9.2]. These establish the role of the (global) product formula in my theory and notably establishes how this provides a (global) arithmetic period mapping. As can be seen from [IUT1, Theorem A, Page 18], [IUT3, Remark 3.9.6, Page 564], Mochizuki asserts various aspects of this point (either implicitly in proofs of [IUT 3, Cor. 3.12] or explicitly). But my approach to this as a period mapping is the most natural formulation.

(3) My preprints are still evolving and as I revisit them, further additions and improvements are definitely to be expected.

0.3

It will be useful to understand the logical dependency of various theorems of [IUT 3] and [IUT 4]. This is not the order in which these results appear in [IUT 3], [IUT 4]. But in my opinion, this is the order in which they should be assimilated.

(1) [IUT 4, Corollary 2.2] Existence of Initial Theta Data. These data are required for construction of the set Θ (discussed below).

(2) [IUT 3, Corollary 3.12]–requires the previous result [IUT 4, Corollary 2.2] for constructing Θ and provides lower bound on the volume $Vol(\Theta)$ of Θ.

(3) [IUT 4, Theorem 1.10]–provides an upper bound on $Vol(\Theta)$.

(4) [IUT 4, Corollary 2.3]–provides the main result (abc-conjecture) of [IUT 1-4].

However, for the ease of comparison between my papers and Mochizuki's, in [Constr. III, IV] I have preserved Mochizuki's original content-wise appearance by putting results of [IUT 3] in [Constr. III] and results of [IUT 4] in [Constr. IV]. But in my opinion, the above ordering is more logical and this is the ordering in which these results *should* be read.

0.4

Important to note that Mochizuki [IUT 4] (or my work) does not directly tackle Szpiro's inequality. Rather one tackles Vojta's inequality. By [Mochizuki 2010, Thm 2.1], Vojta's inequality can be proved by reduction to compactly bounded subsets. To understand the claims made in my paper or [IUT 4], let

$$U = \mathbb{P}^1 - \{0, 1, \infty\}$$

and for a fixed integer $d \geq 1$, let

$$U_d(\overline{\mathbb{Q}}) = U(\overline{\mathbb{Q}})^{\leq d} = (\mathbb{P}^1 - \{0, 1, \infty\})(\overline{\mathbb{Q}})^{\leq d}$$

be the set of algebraic points of degree $\leq d$. The relationship between U and elliptic curves is given by viewing U as j-line for the Legendre family of elliptic curves:

$$U \ni j = j_\lambda$$

where j_λ is the j-invariant of the Legendre elliptic curve

$$C_\lambda : y^2 = x(x-1)(x-\lambda).$$

0.5

I will keep to the general strategy that Mochizuki adopts rather than getting into specifics (for clarity). Mochizuki's strategy for proving Vojta's inequality (of [IUT 4, Thm 1.10]) takes the following shape. We want to prove (on a given compactly bounded subset of $U_d(\overline{\mathbb{Q}})$ whose support contains all primes over $\{2, \infty\}$) that:

$$0 < X \leq Y. \tag{0.5.1}$$

This inequality is proved (both in [IUT 4] and [Constr. IV]) in an indirect fashion while working with the given compactly bounded subset of $U_d(\overline{\mathbb{Q}})$ supported on a finite set of primes containing all primes lying over $\{2, \infty\}$. [Let me remark that even for a compactly bounded set, there is a set of exceptions, but I will not discuss that point here. This is treated out both in [Constr. IV] and [IUT 4].]

Working with a compactly bounded set whose support contains all primes lying over $\{2, \infty\}$ means in particular that $j \in U_d(\overline{\mathbb{Q}})$ lives in a compact subset of

$$(\mathbb{P}^1 - \{0, 1, \infty\})(\mathbb{C}) = \mathbb{C} - \{0, 1\}$$

for all embeddings of $\mathbb{Q}(j) \hookrightarrow \mathbb{C}$. Hence under this assumption, for all embeddings of $\mathbb{Q}(j) \hookrightarrow \mathbb{C}$, the absolute value $|j|_{\mathbb{C}}$ is bounded (from above and from below) and a similar assertion holds for all primes in the support of the compactly bounded subset. The inequality (0.5.1) is established for j-values in the given compactly bounded subset.

0.6

The main idea of [IUT 3, 4] to prove (0.5.1) is to construct a set Θ. The construction of this set Θ requires the existence of Initial Theta Data and this existence itself requires one to work with a compactly bounded subset! Notably

(1) in general, no version of Corollary 3.12 is available without the existence of Initial Theta Data,

(2) and in general Initial Theta Data are available only on a given compactly bounded subset ([IUT 4, Cor 2.2]).

(3) So Corollary 3.12 can be claimed only on a given compactly bounded subset.

(4) Various parameters, which enter Corollary 3.12 and [IUT, Theorem 1.10], require the existence of Initial Theta Data and hence are dependent on this compactly bounded subset (in general).

(5) Especially no version of Corollary 3.12 is available on all of $U_d(\overline{\mathbb{Q}})$.

Now for a fixed compactly bounded subset providing Initial Theta Data, this set Θ is a subset of an adelic object i.e.

$$\Theta = \prod_p \Theta_p \subset \prod_p V_p$$

where, p runs through all places of $\mathbb{Q}$, and V_p is some finite dimensional p-adic vector space.

0.7

Each V_p is in fact a tensor product of some p-adic fields (considered as $\mathbb{Q}_p$-vector spaces). Each V_p is equipped with some p-adic volume form which pays attention to the tensor product structure and is different from the standard volume form. Notably it is not a translation invariant volume i.e. not a Haar measure. Moreover

$$Vol(\Theta) = \prod_p Vol_p(\Theta_p).$$

0.8

But the important point is that Θ is a set of adelic type and the volume of all but finitely many components $Vol_p(\Theta_p) = 1$. In particular, $Vol(\Theta)$ is finite if and only if each Θ_p has a finite volume. Mochizuki works with logarithms of volumes and not volumes. I will simply write $LogVol$ for the natural logarithms of volumes and hence $LogVol_p(\Theta_p) = 0$ for all but a finite number of primes p.

To be absolutely precise, in [IUT 4] or [Constr. IV] one actually works with $Vol(\Theta)^{1/\ell^*}$ (instead of $Vol(\Theta)$) where $\ell^* = \frac{\ell-1}{2}$ for a suitably chosen prime number $\ell \geq 5$. This difference does not affect the present discussion in any way.

Purely for the sake of exposition, I will pretend here that $LogVol_p(\Theta_p) > 0$ when it is non-zero to avoid dealing with signs and sign conventions in Mochizuki's paper (but my paper works the inequalities out carefully avoiding this sort of simplification). [In the current version of [Constr. III, IV] there are some typos, $LogVol(\Theta)$ appears in some place where $|LogVol(\Theta)|$ should. But these typos will be fixed in the next update.]

0.9

On a fixed compactly bounded subset of $U_d(\overline{\mathbb{Q}})$, where one has the Initial Theta Data required for defining Θ, we seek a bound of the form:

$$A \leq LogVol(\Theta) = \sum_p LogVol_p(\Theta_p) \leq A'. \qquad (0.9.1)$$

Note:

(1) This equation (0.9.1) is not the Vojta inequality (0.5.1) we are trying to prove!

(2) The above bounds are not claimed by me on all of $U(\overline{\mathbb{Q}})$ or even on $U_d(\overline{\mathbb{Q}})$ but are claimed only on a given compactly bounded subset. See my additional remarks in 0.12 below for more on this.

0.10

At any rate, for each given compactly bounded subset of $U_d(\overline{\mathbb{Q}})$, by construction, Θ is a set of adelic sort. Hence the global volume of Θ can be bounded both from above and below by summing component volumes for all p. There is no known obstruction to this which I can think of.

0.11

Calculating at each prime p, say we find for suitable real numbers $z_p, z'_p \in \mathbb{R}_{\geq 0}$, that

$$A = \sum_p z_p \leq \sum_p LogVol_p(\Theta_p)$$

and

$$\sum_p LogVol_p(\Theta_p) \leq \sum_p z'_p = A'$$

with $z_p, z'_p = 0$ for all but a finite number of primes while $z_p > 0$ for a finite, non-empty set of primes. Then one has

$$0 < A \leq LogVol(\Theta) = \sum_p LogVol_p(\Theta_p) \leq A'. \tag{0.11.1}$$

Then Mochizuki's assertion in [IUT 4] or [Constr. IV] is that

$$A' = C \cdot A \text{ for some constant } C \in \mathbb{R}, \text{ and} \tag{0.11.2}$$

$$C = Y - X + 1 \text{ for } X, Y \text{ as in (0.5.1).} \tag{0.11.3}$$

Hence Mochizuki deduces from (0.11.1), (0.11.2) and (0.11.3) that

$$C = Y - X + 1 \geq 1$$

and hence deduces the required inequality (on a fixed compactly bounded subset):

$$X \leq Y.$$

The following remarks are important:

(1) As far as I see, one arrives at the equation (0.11.2) only after summing over all the primes.

(2) Note that one is not claiming that $z_p \leq z'_p$ holds for each p and in fact this inequality may not hold for some p while (0.11.1) holds. Of course, if for each prime $z_p \leq z'_p$, then (0.11.1) certainly holds. Local inequalities $z_p \leq z'_p$ will always hold (for all primes) for some curves whose discriminants are close to their conductor. For example the curve 11.a1 in LMFDB has discriminant -11 and conductor 11. In this example, Szpiro's Inequality holds trivially for all $\varepsilon > 0$ with the Szpiro constant 1 for all $\varepsilon > 0$. But for a randomly chosen elliptic curve, $z_p \leq z'_p$ may generally fail to hold for some of the relevant primes.

(3) That is why, the best one can hope here is to compare global (0.11.1) upper and lower bounds on $LogVol(\Theta)$. This is also Mochizuki's point, but he has articulated this quite differently. In [IUT 3] Mochizuki takes a different approach to dealing with this aspect by separating the domain and the codomain of the Θ_{gau}-Link in [IUT 3, Corollary 3.12]. That approach is, of course, available in my work, but in [Constr. III], I have chosen to avoid it for simplicity. Again, importantly, (0.11.1) is not claimed to hold for all elliptic curves or all $U(\overline{\mathbb{Q}})$ or even all $U_d(\overline{\mathbb{Q}})$, but claimed only for a given compactly bounded subset.

Now to address the concerns raised by Peter Scholze regarding [Constr. III, IV]. Scholze has asserted on MathOverFlow that my proof is local i.e. $Vol(\Theta)$ cannot be computed locally and especially that the possible failure of local inequalities $z_p \leq z'_p$ is a weakness of my results. On the other hand the possible failure of local inequalities $z_p \leq z'_p$ is an important aspect of Szpiro's inequality and a feature of the theory. It is precisely because the local inequalities can fail, the next best assertion one can expect to hold is (0.11.1). That is roughly the strategy of [IUT 3, 4] and [Constr. III, IV].

0.12

Some additional comments are in order:

(1) That $Y - X + 1$ appears in the upper bound calculation has nothing to do with component bounds z_p, z'_p.

(2) I do not see any philosophical reason why $C = Y - X + 1$ should appear here. But Mochizuki's calculation of this in [IUT 4] is quite solid and many people including I have independently checked this quite thoroughly.

(3) Especially surprising is the fact that regardless of the choice of a compactly bounded subset, C takes the above shape.

(4) [IUT 4, Thm 1.10] and my [Constr. IV, Thm 6.10.1] works by fixing the compactly bounded set and then establishing the sought (0.5.1) inequality for this subset.

(5) Mochizuki's calculation of the upper bound is local and is in [IUT 4]. The lower bound is asserted by Cor. 3.12 and my observation (but not Mochizuki's) is that the lower bound is also a local calculation (for a fixed compactly bounded subset).

(6) I do not see any obstruction or contradiction here because eq. (0.11.1) is not claiming Szpiro's inequality or abc or Vojta's inequality (0.5.1).