Expectation of binomial random variables

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Here is an alternative to the derivation given in the text, making use of a different trick. Let n be a positive integer and $p \in [0,1]$. Our goal is to calculate the mean of a binomial random variable $X \sim \text{Bin}(n,p)$. That is, we want to evaluate the sum

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}. \tag{1}$$

Here is the trick: consider the function of two real variables

$$f(p,q) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}. \tag{2}$$

Note

$$f(p,q) = (p+q)^n \tag{3}$$

by the Binomial Theorem. In particular, $f(p,1-p) = 1$, which is just a restatement of the fact that the binomial distribution is a probability mass function.

Now, if we differentiate f with respect to p, we get

$$\frac{\partial}{\partial p} f(p,q) = \sum_{k=1}^{n} k \binom{n}{k} p^{k-1} q^{n-k}. \tag{4}$$

(We start the sum at $k=1$ because the $k=0$ term is just q^n, and $\frac{\partial}{\partial p} q^n = 0$.) Thus, we have

$$\begin{aligned} p \frac{\partial}{\partial p} f(p,q) &= p \sum_{k=1}^{n} k \binom{n}{k} p^{k-1} q^{n-k} \\ &= \sum_{k=1}^{n} pk \binom{n}{k} p^{k-1} q^{n-k} \\ &= \sum_{k=1}^{n} k \binom{n}{k} p^{k} q^{n-k}. \end{aligned}$$

If we now set $q = 1 - p$, we get

$$p\Big(\frac{\partial}{\partial p} f(p,q)\Big)_{q=1-p} \quad = \quad \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \quad = \quad E[X]. \tag{5}$$

So, if we can calculate $p\frac{\partial}{\partial p} f(p,q)$, we are done. But we have another expression for $f(p,q)$, namely Eq. (3), which tells us

$$\begin{aligned} p\frac{\partial}{\partial p} f(p,q) &= p\frac{\partial}{\partial p}\Big((p+q)^n\Big) \\ &= pn(p+q)^{n-1}. \end{aligned}$$

Plugging in $q = 1 - p$, we get $E[X] = np$.