1) For this problem, it is useful to know that two events $A$ and $B$ are conditionally independent given a third event $C$ if
\[ P(A \cap B | C) = P(A | C) \cdot P(B | C). \] (1)

(a) Show that if $X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain, then for every $m, n \geq 0$ and $x_i \in S$,
\[
P\left( X_0 = x_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, \quad X_{n+1} = x_{n+1}, \ldots, X_{n+m} = x_{n+m} \mid X_n = x_n \right) \]
\[
= P\left( X_0 = x_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1} \mid X_n = x_n \right) \cdot P\left( X_{n+1} = x_{n+1}, \ldots, X_{n+m} = x_{n+m} \mid X_n = x_n \right).
\]

In words: for a Markov chain, the future and the past are conditionally independent given the present.

(b) Suppose a chain $X_n$ is irreducible. Show that for all $n \geq 0$ and states $x, y, z$ with $y \neq x$, the events $(T_x > n)$ and $(X_{n+1} = z)$ are conditionally independent given $X_n = y$.

2) Let $P$ be a stochastic matrix with the given eigenvalues. Say whether the Markov chain described by $P$ is (i) irreducible, and (ii) converges to equilibrium. Briefly justify.

(a) $\left\{ \pm 1, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \right\}$

(b) $\{1, 1, \pm 0.5\}$

(c) $\{\pm 1, \pm 1/3\}$

(d) $\{1, -1/2, -1/2\}$

3) Let $P$ be an $N \times N$ doubly stochastic matrix, and suppose the corresponding Markov chain $(X_n)$ is irreducible. Show that $(X_n)$ satisfies the detailed balance condition if and only if $P$ is a symmetric matrix.

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1The converse is actually also true, but I'm not asking you to show that here.
2A repeated eigenvalue has the indicated multiplicity, i.e., "1,1" means 1 is a double root of the characteristic equation $\det(P - \lambda I) = 0$. 