Notes on Example 2.1

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I’d like to expand on Example 2.1, because the text has an interesting solution to this standard problem. Here is the mathematical problem: suppose we have two independent exponential random variables $S$ and $T$ with rates $\lambda$ and $\mu$, respectively. What is $E(S \lor T)$? (Recall that for two real numbers $x$ and $y$, “$x \lor y$” means the larger of $x$ and $y$.) The more pedantic approach is to write down the joint PDF

$$f_{ST}(s, t) = \lambda \mu e^{-\lambda s - \mu t}; \quad s, t \geq 0, \quad (1)$$

then compute the expectation

$$E(S \lor T) = \lambda \mu \int_0^\infty \int_0^\infty (s \lor t) e^{-\lambda s - \mu t} \, ds \, dt. \quad (2)$$

The integral can be split into two pieces, according to whether $s < t$ or $s > t$, and evaluated. A more elegant solution was given in the text: we know $S \land T$ (the minimum of $S$ and $T$) is exponential with rate $\lambda + \mu$. Now, if $S < T$ (an event of probability $\lambda / (\lambda + \mu)$), then the expected time for $T$ is again $ET$, since exponential random variables are memoryless. Likewise, if $T < S$, then the expected time for $S$ is $ES$. Putting all this together, we have

$$E(S \lor T) = E(S \land T) + P(S < T) \cdot ET + P(T < S) \cdot ES \quad (3a)$$

$$= \frac{1}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\mu} + \frac{\mu}{\lambda + \mu} \cdot \frac{1}{\lambda}. \quad (3b)$$

This is a very slick argument, but you may feel it’s a little too slick. The purpose of this note is to expand on it, so it doesn’t look so slick.
Let’s start with $E(S \lor T)$, and condition on which exponential waiting time is smaller:

$$E(S \lor T) = E(S \lor T \mid S < T) \cdot P(S < T) + E(S \lor T \mid T < S) \cdot P(T < S) \quad (4a)$$

$$= E(T \mid S < T) \cdot P(S < T) + E(S \mid T < S) \cdot P(T < S). \quad (4b)$$

Let’s now compute $E(T \mid S < T)$, intentionally writing things in more abstract notation so that the key step is more apparent:

$$E(T \mid S < T) \cdot P(S < T) \quad (5a)$$

$$= \int_0^\infty \int_0^\infty t \cdot f_S(s) f_T(t) \, dt \, ds \quad (5b)$$

$$= \int_0^\infty f_S(s) \int_s^\infty t \cdot f_T(t \mid T > s) P(T > s) \, dt \, ds \quad (5c)$$

$$= \int_0^\infty f_S(s) P(T > s) \left( \int_0^s t \cdot f_T(t \mid T > s) \, dt \right) \, ds \quad (5d)$$

where

$$f_T(t \mid T > s) = \begin{cases} \frac{f_T(t)}{P(T > s)}, & t > s \\ 0, & t < s \end{cases} \quad (6)$$

is the conditional PDF of $T$ given $T > s$ and $(*)$ is the corresponding conditional expectation $E(T \mid T > s)$. Observe now that

$$f_T(t \mid T > s) = \frac{d}{dt} P(T \leq t \mid T > s) \quad (7a)$$

$$= \frac{d}{dt} (1 - P(T > t \mid T > s)) \quad (7b)$$

$$= \frac{d}{dt} (1 - P(T > t - s)) \quad (7c)$$

$$= \frac{d}{dt} P(T \leq t - s). \quad (7d)$$

The crucial step was from Eq. (7b) to (7c), where we used the memory less property of the exponential distribution. From the above, we get

$$f_T(t \mid T > s) = f_T(t - s). \quad (8)$$
(This could have obtained with a couple lines of algebra, but then it would not
be so clear where we used the memoryless property!) Substituting this into (*)
in Eq. (5), we get

\[ E(T \mid T > s) = \int_s^\infty t \cdot f_T(t - s) \, dt \]  
(9a)

\[ = \int_0^\infty (t + s) \cdot f_T(t) \, dt \]  
(9b)

\[ = s + ET. \]  
(9c)

(This formula can also be anticipated by the same argument used to derive
Eq. (3).) So Eq. (5) becomes

\[ E(T \mid S < T) \cdot P(S < T) = \int_0^\infty f_S(s) \, P(T > s) \, (ET + s) \, ds. \]  
(10)

Finally, plugging in \( f_S(s) = \lambda e^{-\lambda s} \) and \( P(T > s) = e^{-\mu s} \), we get

\[ E(T \mid S < T) = \lambda \int_0^\infty e^{-(\lambda + \mu)s} (1/\mu + s) \, ds \]  
(11a)

\[ = \frac{\lambda}{\lambda + \mu} \left( \frac{1}{\mu} + \frac{1}{\lambda + \mu} \right). \]  
(11b)

Incidentally, this means

\[ E(T \mid S < T) = \frac{1}{\mu} + \frac{1}{\lambda + \mu}, \]  
(12)

which we could also have obtained using the “slick” argument.

Similarly,

\[ E(S \mid T < S) = \frac{1}{\lambda} + \frac{1}{\lambda + \mu}, \]  
(13)

so

\[ E(S \lor T) = \frac{\lambda}{\lambda + \mu} \left( \frac{1}{\mu} + \frac{1}{\lambda + \mu} \right) + \frac{\mu}{\lambda + \mu} \left( \frac{1}{\lambda} + \frac{1}{\lambda + \mu} \right), \]  
(14)

which is equivalent to Eq. (3b).