Today’s agenda:
- A summary of key theorems regarding stationary distributions
- The Perron-Frobenius Theorem.
- A sketch of the Convergence Theorem for 2-state chains using the Perron-Frobenius Theorem.

Stationary distributions

First, there seems to be some confusion about basic definitions:
- A probability distribution on the state space $S$ is \textit{stationary} if for all $x \in S$,
  \[
  \sum_{y \in S} \pi(y)p(y, x) = \pi(x). \tag{1}
  \]

- A Markov chain satisfied detailed balance with respect to a probability distribution $\pi$ on the state space $S$ if for all $x, y \in S$,
  \[
  \pi(y)p(y, x) = \pi(x)p(x, y). \tag{2}
  \]
  Note that detailed balance is strictly stronger than stationarity: it implies stationarity but stationarity does not imply detailed balance.

- A Markov chain is said to \textit{converge to equilibrium} if
  \[
  p^n(x, y) \to \pi(y) \tag{3}
  \]
  as $n \to \infty$. Equivalently, a chain converges to equilibrium if for any initial distribution,
  \[
  P(X_n = y) \to \pi(y). \tag{4}
  \]

Here is a \textbf{summary of key theorems} from the text.
Some of the theorems can be stated a bit more generally without much effort. For example, the assumptions of irreducibility and recurrence in Theorem 1.20 are not really necessary:
Theorem (1.20').
\[ \lim_{n \to \infty} \frac{N_n(y)}{n} = \begin{cases} \frac{1}{E_x \tau_y}, & T_y < \infty \\ 0, & T_y = \infty. \end{cases} \] (5)

Proof. If \( T_y = \infty \) then \( N_n(y) = 0 \), so the conclusion is trivial.
If \( T_y < \infty \) (so the chain does reach \( y \) at some point), and \( y \) is recurrent, then the proof of Theorem 1.20 applies. (Check it!)
Finally, if \( y \) is transient, then
\[
P_y(N(y) = \infty) = \lim_{k \to \infty} P_y(N(y) \geq k) = \lim_{k \to \infty} \rho_{xy}^{k-1} = 0. \] (6c)

So \( N(y) < \infty \) with probability 1. Since \( N_n(y) \leq N(y) \) for all \( n \), the conclusion follows. \( \square \)

Another theorem that generalizes slightly is Theorem 1.19. To avoid complications, here I just state a result for finite state spaces.

Theorem (1.19'). If \( |S| < \infty \) and \( S \) is a disjoint union \( T \cup R \), where \( T \) is the set of all transient states and \( R \) is closed, irreducible, and aperiodic. Then \( p^n(x, y) \to \pi(y) \) for all \( x, y \in S \).

Proof. We go through the different cases:
If \( x, y \in R \), then the result is just Theorem 1.19.
If \( x \in R \) and \( y \in T \), then from the proof of the Decomposition Theorem we know that \( p^n(x, y) = 0 \).
If \( x, y \in T \) and \( x \) does not communicate with \( y \), then \( p^n(x, y) = 0 \) for all \( n \).
If \( x, y \in T \) and \( x \) does communicate with \( y \), then there is an \( m \) such that \( p^m(x, y) > 0 \). And \( p^{m+n}(x, y) \geq P(x, y)p^n(y, y) \).

\[
\sum_{n=1}^{\infty} p^n(x, y) = \sum_{n=1}^{\infty} P_x(X_n = y) = E_x \sum_{n=1}^{\infty} \mathbb{1}_{(X_n = y)} = E_x N(y) = E_x(N(y)|T_y < \infty) \cdot \rho_{xy} + E_x(N(y)|T_y = \infty) \cdot (1 - \rho_{xy}) = E_y N(y) \cdot \rho_{xy}. \] (7e)

(The next-to-last line uses the Markov property.) Since \( y \) is transient, we know \( E_y N(y) < \infty \), so \( p^n(x, y) \to 0 \).

\( ^{1} \)I said this incorrectly in lecture today. Duh!
Finally, suppose $x \in T$ and $y \in R$. We know from the Decomposition Theorem that there is some $z \in R$ such that $x \to z$ and $\rho_{xz} = 0$. Since $R$ is recurrent (because it is closed, irreducible, and finite) we have $\rho_{zy} = 1$, and $x \to y$ as well. As in Eq. (7), we have

$$P_x(X_n = y) = \sum_{k=1}^{\infty} P_x(X_n = y | T_y = k) \cdot P_x(T_y = k)$$  \hspace{1cm} (8)

For each $k$, we have

$$P^n(x, y) = P_x(X_n = y | T_y = k) = P_y(X_{n-k} = y)$$  \hspace{1cm} (9a)

$$\lim_{n \to \infty} P^n(x, y) = \lim_{n \to \infty} P_y(X_{n-k} = y) = \pi(y)$$  \hspace{1cm} (10a)

by Theorem 1.19, and

$$\lim_{n \to \infty} P_x(X_n = y) = \pi(y) \sum_{k=1}^{\infty} P_x(T_y = k) = \pi(y) \cdot \rho_{xy}.$$  \hspace{1cm} (11a)

The last thing we need is $\rho_{xy} = 1$. Heuristically, this is because $x \in T$, so eventually it reaches a recurrent state $z \in R$. Since $R$ is closed and irreducible (and hence recurrent), $\rho_{zy} = 1$ for any $z$. So $\rho_{xy} = 1$.

Perron-Frobenius Theorem

**Theorem.** Let $P$ be a stochastic matrix, and suppose all states are recurrent. Then

1. $1$ is an eigenvalue of $P$, and it has a left eigenvector $\pi$ with all nonnegative entries. For all eigenvalues $\lambda$ of $P$, $|\lambda| \leq 1$.

2. If $P$ is irreducible, then the eigenvalue $1$ has multiplicity $1$, i.e., there is a unique (left) eigenvector $\pi$ such that $\pi \cdot P = \pi$, $\pi(x) > 0$ for all $x$, and $\sum x \pi(x) = 1$. More generally, if the chain has $k$ closed irreducible blocks, then the eigenvalue $1$ has multiplicity $k$, and there are exactly $k$ linearly independent stationary distributions.

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2 This is not rigorous, because I do not justify exchanging the limit with the infinite sum. But it hopefully gives you an idea for why this is true – think about what the chain is doing!

3 This can be made more airtight with a little effort.
3) If $P$ is irreducible with period $d$, then $P$ has exactly $d$ eigenvalues $\lambda$ with $|\lambda| = 1$, corresponding to the roots of $\lambda^d = 1$, i.e., they are the $d$th roots of unity. All other eigenvalues of $P$ have absolute value $< 1$.

4) If $P$ is irreducible and aperiodic, then all eigenvalues $\lambda \neq 1$ have $|\lambda| < 1$.

5) If $P$ is irreducible and satisfies detailed balance with respect to the (unique) stationary distribution $\pi$, then all eigenvalues are real.

Notes:

1) See [this snapshot of the Wikipedia page](#) for more information.

2) Here are [some examples](#) and the associated [Jupyter notebook](#).