Today's agenda:
- A summary of key theorems regarding stationary distributions
- The Perron-Frobenius Theorem.
- A sketch of the Convergence Theorem for 2-state chains using the Perron-Frobenius Theorem.

Stationary distributions

First, there seems to be some confusion about basic definitions:
- A probability distribution on the state space $S$ is stationary if for all $x \in S$,
  \[ \sum_{y \in S} \pi(y)p(y, x) = \pi(x). \]  
  \hspace{1cm} (1)

- A Markov chain satisfied detailed balance with respect to a probability distribution $\pi$ on the state space $S$ if for all $x, y \in S$,
  \[ \pi(y)p(y, x) = \pi(x)p(x, y). \]  
  \hspace{1cm} (2)

Note that detailed balance is strictly stronger than stationarity: it implies stationarity but stationarity does not imply detailed balance.
- A Markov chain is said to converge to equilibrium if
  \[ p^n(x, y) \to \pi(y) \]  
  as $n \to \infty$. Equivalently, a chain converges to equilibrium if for any initial distribution,
  \[ P(X_n = y) \to \pi(y). \]  
  \hspace{1cm} (4)

Here is a summary of key theorems from the text.
- Some of the theorems can be stated a bit more generally without much effort. For example, the assumptions of irreducibility and recurrence in Theorem 1.20 are not really necessary:
Theorem (1.20').

\[
\lim_{n \to \infty} \frac{N_n(y)}{n} = \begin{cases} \frac{1}{E_y T_y}, & T_y < \infty \\ 0, & T_y = \infty \end{cases}
\]

Proof. If \( T_y = \infty \) then \( N_n(y) = 0 \), so the conclusion is trivial.

If \( T_y < \infty \) (so the chain does reach \( y \) at some point), and \( y \) is recurrent, then the proof of Theorem 1.20 applies. (Check it!)

Finally, if \( y \) is transient, then \( 1 P_y \left( N(y) = \infty \right) = \lim_{k \to \infty} P_y \left( N(y) \geq k \right) = \lim_{k \to \infty} \rho_{xy}^{k-1} = 0 \). (6c)

So \( N(y) < \infty \) with probability 1. Since \( N_n(y) \leq N(y) \) for all \( n \), the conclusion follows. \( \square \)

Another theorem that generalizes slightly is Theorem 1.19. To avoid complications, here I just state a result for finite state spaces.

Theorem (1.19'). If \( |S| < \infty \) and \( S \) is a disjoint union \( T \cup R \), where \( T \) is the set of all transient states and \( R \) is closed, irreducible, and aperiodic. Then \( p^n(x, y) \to \pi(y) \) for all \( x, y \in S \).

Proof. We go through the different cases:

If \( x, y \in R \), then the result is just Theorem 1.19.

If \( x \in R \) and \( y \in T \), then from the proof of the Decomposition Theorem we know that \( p^n(x, y) = 0 \).

If \( x, y \in T \) and \( x \) does not communicate with \( y \), then \( p^n(x, y) = 0 \) for all \( n \).

If \( x, y \in T \) and \( x \) does communicate with \( y \), then there is an \( m \) such that \( p^m(x, y) > 0 \). And \( p^{m+n}(x, y) \geq p^m(x, y)p^n(y, y) \).

\[
\sum_{n=1}^{\infty} p^n(x, y) = \sum_{n=1}^{\infty} P_x(X_n = y) = E_x \sum_{n=1}^{\infty} \mathbb{I}(x_n = y) = E_x N(y) = E_x \left( N(y) | T_y < \infty \right) \cdot \rho_{xy} + E_x \left( N(y) | T_y = \infty \right) \cdot (1 - \rho_{xy}) = E_y N(y) \cdot \rho_{xy}. \]

(The next-to-last line uses the Markov property.) Since \( y \) is transient, we know \( E_y N(y) < \infty \), so \( p^n(x, y) \to 0 \).

\footnote{I said this incorrectly in lecture today. Duh!}
Finally, suppose $x \in T$ and $y \in R$. We know from the Decomposition Theorem that there is some $z \in R$ such that $x \to z$ and $\rho_{zx} = 0$. Since $R$ is recurrent (because it is closed, irreducible, and finite) we have $\rho_{zy} = 1$, and $x \to y$ as well. As in Eq. (7), we have

$$P_x(X_n = y) = \sum_{k=1}^{\infty} P_x(X_n = y | T_y = k) \cdot P_x(T_y = k) \quad (8)$$

For each $k$, we have

$$p^n(x, y) = P_x(X_n = y | T_y = k) = P_y(X_{n-k} = y) \quad (9a)$$

for $n > k$, by the Markov property. So

$$\lim_{n \to \infty} p^n(x, y) = \lim_{n \to \infty} P_y(X_n = y) = \pi(y) \quad (10a)$$

by Theorem 1.19, and

$$\lim_{n \to \infty} P_x(X_n = y) = \pi(y) \sum_{k=1}^{\infty} P_x(T_y = k) = \pi(y) \cdot \rho_{xy}. \quad (11a)$$

The last thing we need is $\rho_{xy} = 1$. Heuristically, this is because $x \in T$, so eventually it reaches a recurrent state $z \in R$. Since $R$ is closed and irreducible (and hence recurrent), $\rho_{zy} = 1$ for any $z$. So $\rho_{xy} = 1$. \(\square\)

**Perron-Frobenius Theorem**

**Theorem.** Let $P$ be a stochastic matrix. Then

1) $1$ is an eigenvalue of $P$, and it has a left eigenvector $\pi$ with all nonnegative entries.

2) $|\lambda| \leq 1$ for all eigenvalues $\lambda$ of $P$.

3) If $P$ is irreducible, then the eigenvalue $1$ has multiplicity $1$, i.e., there is a unique (left) eigenvector $\pi$ such that $\pi \cdot P = \pi$, $\pi(x) > 0$ for all $x$, and $\sum_x \pi(x) = 1$.

4) More generally, if the chain has $k$ closed irreducible blocks, then the eigenvalue $1$ has multiplicity $k$, and there are $k$ linearly independent stationary distributions.

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2This is not rigorous, because I do not justify exchanging the limit with the infinite sum. But it hopefully gives you an idea for why this is true – think about what the chain is doing!

3This can be made more airtight with a little effort.
5) If \( P \) is irreducible with period \( d > 1 \), then \( P \) has exactly \( d \) eigenvalues \( \lambda \) with \( |\lambda| = 1 \). The eigenvalues all satisfy \( \lambda^d = 1 \), i.e., they are the \( d \)th roots of unity. All other eigenvalues of \( P \) have absolute value \( < 1 \).

6) If \( P \) is irreducible and aperiodic, then all eigenvalues \( \lambda \neq 1 \) have \( |\lambda| < 1 \).

7) If \( P \) is irreducible and satisfies detailed balance with respect to the (unique) stationary distribution \( \pi \), then all eigenvalues are real.

Notes:

1) See this snapshot of the Wikipedia page for more information.

2) Here are some examples and the associated Jupyter notebook.