Today, we started Chapter 2 (hooray!). I mainly reviewed exponential random variables and their various properties, including:

1) If $T_1, \ldots, T_n$ are independent and $T_i \sim Exp(\lambda_i)$, then $\min(T_1, \ldots, T_n) \sim Exp(\sum_i \lambda_i)$. Moreover, if $I$ is the index of the minimum, i.e., $T_I = \min(T_1, \ldots, T_n)$, then $I$ and $\min(T_1, \ldots, T_n)$ are independent.

2) If $T_1, \ldots, T_n$ are independent and $T_i \sim Exp(\lambda)$ for all $i$, then $\sum_i T_i$ is a gamma$(n, \lambda)$ random variable (see text).

I basically followed the proofs in the text, with the exception of the independence of $I$ and $\min(T_1, \ldots, T_n)$. Instead of the proof in the book, I sketched how for all $i \in \{1, \ldots, n\}$ and $0 \leq a \leq b$, we have

$$P(I = i) \cap (a \leq \min(T_1, \ldots, T_n) \leq b) = P(I = i) \cdot P(a \leq \min(T_1, \ldots, T_n) \leq b).$$

(1)

This is enough to imply that for all events $E \subset \{1, \ldots, n\}$ and $F \subset [0, \infty)$, we have $P(I \in E) \cap (\min(T_1, \ldots, T_n) \in F)) = P(I \in E) \cdot P(\min(T_1, \ldots, T_n) \in F)$.

**Addendum, April 9, 2020.** I mentioned in passing that the exponential distribution is essentially the only one with the memoryless property. Here is a partial “proof.” (I’m not stating the underlying assumptions.) Suppose $T$ is a continuous random variable such that $P(T > 0) = 1$ and it has the memoryless property, i.e.,

$$P(T > t + s | T > s) = P(T > t)$$

(2)

for all $s, t \geq 0$. Let

$$G(t) = P(T > t).$$

(3)
Then

\[ G(t + s) = P(T > t + s) \]
\[ = P(T > t + s, T > s) \]
\[ = P(T > t + s | T > s) P(T > s) \]
\[ = P(T > t) P(T > s) \]
\[ = G(t) G(s). \]

So

\[ G(t + s) - G(t) = G(t)(G(s) - 1) \] (4)

Dividing both sides by \( s \) and letting \( s \to 0 \) yields

\[ G'(t) = G(t) \cdot G'(0). \] (5)

Since \( G(t) \) is decreasing (or at least non-increasing), \( G'(0) \leq 0 \). Let \( \lambda = -G'(0) \)
yields \( G'(t) = -\lambda G(t) \), so that \( G(t) = e^{-\lambda t} \).