Math 583A Fall 2011 Problem Set #1

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Due Tuesday, 9/6

1. Consider the diffusion equation
$$u_t = Du_{xx} \tag{1}$$
on the real line, with diffusion constant $D > 0$ and vanishing boundary conditions at infinity.

   (a) Show that setting $u(x,t) = U(x^2/Dt)$ yields an ODE for $U(\xi)$, where $\xi = x^2/Dt$. *(Correction: should be $\xi = x/\sqrt{Dt}$)*

   (b) Solve the ODE subject to the boundary conditions $U(-\infty) = 0$ and $U(+\infty) = 1$. Express your answer in terms of the function $\Phi(a) = \int_{-\infty}^{a} e^{-\frac{1}{2}x^2}\,dx$. *(Notice these solutions are invariant under the scaling $\tilde{x} = Lx$, $\tilde{t} = L^2 t$ — same as Eq. (1) itself!)*

2. For each of the following equations, (i) nondimensionalize it, and (ii) say how many nondimensional parameters you found.

   (a) $u_{tt} = c^2 u_{xx} + au + bu^2$; boundary conditions: $\lim_{x\to\pm\infty} u(x,t) = 0$

   (b) $u_{tt} = c^2 u_{xx} + au + bu^2 + ku^3$; boundary conditions: $\lim_{x\to\pm\infty} u(x,t) = 0$

   (c) $u_{tt} = c^2 u_{xx} + au + bu^2 + ku^3 + mu^4$; boundary conditions: $u(0,t) = u(L,t) = 0$ for all t

3. Consider the PDE
$$u_t + uu_x = \nu u_{xx} \ , \tag{2}$$
where $x \in \mathbb{R}$, $t \in [0,\infty)$, and $\nu > 0$ is a parameter, with boundary conditions
$$\lim_{x\to-\infty} u(x,t) = U_1 \ , \qquad \lim_{x\to+\infty} u(x,t) = U_2 \ . \tag{3}$$

   (a) Let u be a solution of Eq. (2). For any two numbers $a < b$, define $q(t) = \int_a^b u(x,t)\,dx$. Show that $\dot{q}(t) = j_u(b,t) - j_u(a,t)$ for some function j_u, which you can express in terms of u and its derivatives.

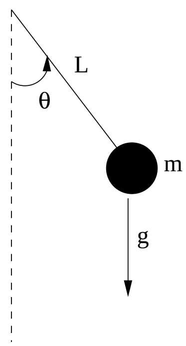
   (As you may know from your physics courses, this says that Eq. (2) is a "conservation law," i.e., the function u can be viewed as representing the spatial distribution of a physically conserved quantity. The quantity $j_u(x,t)$ measures the flux *of conserved quantities at location x and time t.)*

   (b) For $U_1 > 0$ and $U_2 = 0$, find the traveling wave solution for Eq. (2).

   (c) What feature of the traveling wave depends on ν? What other important feature does *not* depend on ν (and what does it depend on)? What happens if we send $\nu \to 0$, and what PDE should the resulting traveling wave satisfy?

   (This is an example of a "weak solution" of a PDE.)

4. Consider the classical pendulum with length L, mass m, and subjected to gravity:



(a) Write down a differential equation of the form $\ddot{\theta} = F(\theta)$ for the pendulum. You should assume there is no friction. Also, do *not* use a small-angle approximation, i.e., your answer should be *nonlinear*.

(b) Without rescaling the equation, say how many nondimensional paramters you expect to have after rescaling. Why?

(c) Nondimensionalize the equation. How many free parameters remain after nondimensionalization? What are the scales you found?

(d) Draw the phase portrait for this system.

(e) Suppose we now add a term $-\alpha\dot{\theta}$, $\alpha > 0$, to the right hand side of your equation; such a term represents friction in the pendulum. Find all the equilibria and classify their linear stability.

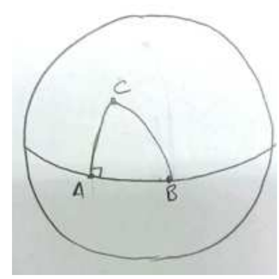
5. (a) Suppose we are given a right triangle. Let θ be one of the non-90° angles and a the length of the ~~adjacent leg~~ hypotenuse.[1] Using dimensional analysis and basic geometry, show that the area $F(a, \theta)$ of the triangle must have the form $a^2 f(\theta)$.

   Notes: (i) You do not have to prove every geometric fact you use, but you must state explicitly all geometric assumptions. *(ii) You can easily do this using trigonometry, and in the process find f. That is not what this problem asks for; a solution using trigonometry will not be given credit.*

   (b) (2) Using the fact above and a little bit of geometry, prove the Pythagorean Theorem. What assumption did you need to make about f?

   (c) (1) Now let $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Given two points $A, B \in S^2$, we define the *minimal geodesic* to be the shortest circular arc on the sphere that connects A and B. Given three distinct points A, B, and C, we can define a "triangle" on S^2 to be the union of the 3 minimal geodesics connecting them. (See picture.) The notion of a "right angle" on S^2 should be intuitive, so we can make sense of "right triangles on S^2."

   Using the above definitions, give an explicit example of a right triangle on S^2 for which the Pythagorean Theorem fails, i.e., $\ell(AB)^2 + \ell(BC)^2 \neq \ell(AC)^2$, where $\ell(\cdot)$ denotes arclength.



   (d) Explain why your proof from Part (b) does not apply to right triangles on the sphere.

(Another fun problem[2] to think about: From the dimensional / scaling point of view, why should the Pythagorean Theorem be approximately true for small triangles on a sphere?)

[1]It doesn't really matter which side we pick, but we need to be consistent, and hypotenuse is a bit more natural.
[2]You do not need to turn in this part.