Math 583A Fall 2011 Problem Set #4

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Due ~~Thursday, 10/13~~ **end of day on Friday, 10/14**

Last revised: 2011.10.12

1. (a) Using principal value integrals, show that for k real,

$$\int_0^\infty \frac{\cos(kx) - 1}{x^2}\, dx = -\frac{\pi}{2}|k|$$

(Note that $f(z) = (e^{ikz} - 1)/z^2$ has a simple pole at $z = 0$.)

(b) Let $k = 2$. Deduce

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\, dx = \frac{\pi}{2}\ .$$

2. Use a semicircular contour enclosing the left half plane (with a suitable "keyhole" contour), show that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z}}\, dz = \frac{1}{\sqrt{\pi t}}$$

for $a, t > 0$. (This is the inverse Laplace transform of $1/\sqrt{z}$.)

3. Evaluate *using contour integrals:*

(a) $\int_0^\infty \frac{(\log x)^2}{x^2+1}\, dx$

(b) $\int_0^1 \sqrt{x(1-x)}\, dx$

Hint: Use a "dog-bone" contour, but be careful with the circular arc.

4. (a) Let $f(z) = (z - z_0)^k$, where k is an integer. Show that

$$\int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i k$$

for any simple closed contour enclosing z_0 . (Note that if $k > 0$, f has a single zero of order k at z_0 , and if $k < 0$, f has a pole of order k at z_0 .)

(b) Prove the *argument principle:* for any meromorphic f and any simple close contour C on which f has no zeros or poles,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = N - P\ ,$$

where N is the number of zeros of f enclosed by C, and P the number of poles. (Zeros and poles are counted with multiplicity, i.e., zeros of order 2 are counted twice, etc.)

(c) How many solutions of $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? (Suggestion: Apply the argument principle to a large half disc, and ~~sketch the image of the imaginary axis~~ *show that the contribution of the imaginary axis is 0.*

Here's one way to do that:

(i) sketch the image of $i\mathbb{R}$ under the mapping f ; then

(ii) show that one can pick a branch of $\log(z)$ so the branch cut doesn't intersect $f(i\mathbb{R})$.

Check that this means the Fundamental Theorem of Calculus applies.)