Math 322 Sect. 2
Homework #12

Due Wed. 4/15/15

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The analogy between Fourier series representations and coordinates relative to orthogonal bases suggests various interesting and useful identities. Here is one such identity. **Please write up #1 & 2 and turn them in. #3 is optional: do it if you are curious, and whenever you have time.**

1. Let x be any vector in $\mathbb{C}^n$ and $(v_1, \cdots, v_n)$ be an orthogonal basis of $\mathbb{C}^n$ with the property that $||v_n|| = 2$ for all n. Let $(c_1, \cdots, c_n$ denote the coordinates of x relative to the basis $(v_1, \cdots, v_n)$. Show that $||x||^2 = K \sum_{k=1}^{n} |c_k|^2$ for some constant K. What is the value of K? *Hint:* $||x||^2 = (x, x)$*; use this and the orthogonality of the basis.*

2. Let f and g be two functions, and define (as we did in class)
$$(f, g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx. \tag{1}$$
Define $||f|| = \sqrt{(f, f)}$, and also
$$v_n(x) = e^{inx} \ . \tag{2}$$
As we discussed in class on Monday, the functions v_n are "orthogonal" to each other in the sense that
$$\begin{aligned}(v_m, v_n) &= \int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}}\, dx \\ &= \int_{-\pi}^{\pi} e^{imx}e^{-inx}\, dx \\ &= \begin{cases} 0, & m \neq n \\ 2\pi, & m = n \end{cases}\end{aligned}$$
Using this, show that
$$||f||^2 = K \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{3}$$
for some constant K. What is the value of K?

   (In many physical problems, $||f||^2$ can be interpreted as the energy per unit time contained in a wave, and Eq. (3) shows that the energy can be expressed in terms of the Fourier coefficients directly.)

3. Let f be the function
$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \tag{4}$$
We showed in class that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with
$$c_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{i\pi n}, & n \text{ odd} \end{cases} \tag{5}$$
Apply the two sides of Eq. (3) to the Fourier series $f(x)$ above to find an explicit expression for $\sum_{n=-\infty}^{\infty} |c_n|^2$.