Math 322 Sect. 2
“Homework” #15

Due Fri. 5/8/15

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These are practice problems for Fourier transforms. I will not collect these, but will post solutions before the final.

Note the first one is the example from class today (Monday 5/4).

1. Let g be the function

$$g(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & -1 \le x \le 0 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

   (a) Sketch g.

   (b) Let

$$f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

   Express g in terms of f via scaling, shifting, and linear combinations. *Hint: let g_+ be the “right half” of g, and g_- the left half. Show that each of g_+ and g_- can be obtained from f by scaling, then shifting. Or shifting, then scaling – you can do it in either order.*[1]

   (c) Find $\widehat{g}(\omega)$.[2]

   (d) Try to do the same for

$$g(x) = \begin{cases} 2, & 1 \le x \le 2 \\ -3, & -1 \le x \le 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

2. Consider the differential equation

$$-u''(x) + u(x) = f(x) \tag{4}$$

   where $-\infty < x < \infty$ and we assume u, f have well-defined Fourier transforms.

   (a) Find $\widehat{u}(\omega)$ in terms of $\widehat{f}(\omega)$.

   (b) By taking inverse Fourier transforms, express $u(x)$ as a convolution $(K * f)(x)$. What is the function K? *Hint: see the solution to Problem 4 on Exam 3.*

[1] If you scale, then shift, you would get $g_+(x) = f(2(x - \frac{1}{2}))$. If you shift, then scale, you get $g_+(x) = f(2x - 1)$.

[2] As an example, if you get g_+ from f by shifting then scaling, you would have $g_+(x) = f(2x-1)$. The two operations can be represented symbolically as

$$g_+ = Scale(Shift(f)).$$

From this, we have

$$\begin{aligned} \mathcal{F}(g_+)(\omega) &= \mathcal{F}(Scale(Shift(f)))(\omega) \\ &= \frac{1}{2}\mathcal{F}(Shift(f))(\omega/2) \\ &= \frac{1}{2}e^{-i\omega/2}\mathcal{F}(f)(\omega/2). \end{aligned}$$

3. The Fourier transform of

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{\sin(x)}{x} \tag{5}$$

is

$$\widehat{f}(\omega) = \begin{cases} 1, & -1 \leq \omega \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

(You can derive this by noticing that the Fourier transform of f is the same as its inverse Fourier transform, because f is even.) Assuming this fact, answer the following questions:

(a) Let $g(x)$ be a function such that $\widehat{g}(\omega) = 0$ for $|\omega| < 1$. What can you say about the Fourier transform of $f * g$?

(b) Let $g(x)$ be a function such that $\widehat{g}(\omega) = 0$ for $|\omega| > 1$. What can you say about the Fourier transform of $f * g$?