Covariance and independence: an example

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This is the last example from class today. Suppose the random variables X, Y are uniformly distributed over the disc $D = \{(x, y) \mid x^2 + y^2 = 1\}$ of radius 1, so that their joint pdf is

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases} \tag{1}$$

Then

$$E[XY] = 0 \tag{2}$$

even though (as was shown in class last week) X and Y are independent.

There are a few different ways to show this. The first is to evaluate the integral:

$$\begin{aligned} \mathrm{Cov}(X,Y) = E[XY] \\ &= \frac{1}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\ dy\ dx \\ &= \frac{1}{\pi} \int 0^1 \int_0^{2\pi} r\cos(\theta) \cdot r\sin(\theta) \cdot r\ d\theta\ dr. \end{aligned}$$

This is straightforward to evaluate, and gives 0.

A second, better (IMO) method is the following: observe that $f(x, y) = f(-x, y)$ for all (x, y), since if $(x, y) \in D$ then we have $f(x, y) = f(-x, y) = 1/\pi$, and if $(x, y) \notin D$ then we just have $f(x, y) = f(-x, y) = 0$. This means

$$\begin{aligned} \mathrm{Cov}(X,Y) = E[XY] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\ f(x,y)\ dy\ dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\ f(-x,y)\ dy\ dx. \end{aligned}$$

We now substitute $u = -x$, so that $du = -dx$ and we have

$$-\int_{\infty}^{-\infty} \int_{-\infty}^{\infty} (-u) \cdot y\ f(u,y)\ dy\ du. \tag{3}$$

Reverse the limits on the outside integral, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-u)\cdot y\ f(u,y)\ dy\ du = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\cdot y\ f(u,y)\ dy\ du.$$

Except for using a different dummy variable for integration, the integral in the last line is just $E[XY]$ again. So the above shows $E[XY] = -E[XY]$. This can only be the case if $E[XY] = 0$.

The reason I like this second proof is it illustrates how symmetries in a problem can lead to random variables being uncorrelaed. In many applications, random variables that are not independent are nevertheless uncorrelated due to symmetries in the problem.