Sums of independent continuous random variables

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Let X and Y be jointly continuous random variables with joint probability density function (PDF) $f_{XY}(x,y)$. The *conditional cumulative distribution function (CDF)* is

$$F_{X|Y}(x|y) \;=\; P(X \leqslant x|Y=y) \;=\; \int_{-\infty}^{x} f_{X|Y}(x|y)\,dx \tag{1}$$

where $f_{X|Y}(x|y) = f_{XY}(x,y)/f_Y(y)$ is the conditional PDF of X given $Y=y$.

Fact 1. As discussed in class, one can express the (unconditional) CDF of X by conditioning on $Y=y$:

$$F_X(x) \;=\; P(X \leqslant x) \;=\; \int_{-\infty}^{\infty} P(X \leqslant x|Y=y)\, f_Y(y)\,dy. \tag{2}$$

One way to see this is to start with

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f_Y(y)\,dy. \tag{3}$$

Integrating both sides from $-\infty$ to x gives

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du \tag{4a}$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X|Y}(u|y)\, f_Y(y)\,dy\,du \tag{4b}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{X|Y}(u|y)\, f_Y(y)\,du\,dy \tag{4c}$$

$$= \int_{-\infty}^{\infty} f_Y(y)\left(\int_{-\infty}^{x} f_{X|Y}(u|y)\,du\right)dy \tag{4d}$$

$$= \int_{-\infty}^{\infty} f_Y(y) F_{X|Y}(x|y)\,dy. \tag{4e}$$

(From (a) to (b) I just changed the order of integration. From (b) to (c), we can pull the marginal PDF $f_Y(y)$ out from the integral because it does not depend on x. From (c) to (d) we use the fact that the expression inside the parentheses is the conditional PDF.)

Fact 2. The independence of X and Y means $f_{X|Y}(x|y) = f_X(x)$. Integrating from $-\infty$ to x gives us

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(u|y)\,du \tag{5a}$$

$$= \int_{\infty}^{x} f_X(u)\,du \tag{5b}$$

$$= F_X(x) \tag{5c}$$

So

$$P(X \leqslant x|Y = y) = P(X \leqslant x) \tag{6}$$

for all possible values of x and y if (and only if) X and Y are independent.

Suppose now X and Y are jointly continuous and independent. Then Fact 1 (Equation 2) gives us

$$F_Z(z) = P(X + Y \leqslant z) \tag{7a}$$

$$= \int_{-\infty}^{\infty} P(X + Y \leqslant z|Y = y) f_Y(y)\,dy \tag{7b}$$

$$= \int_{-\infty}^{\infty} P(X + y \leqslant z|Y = y) f_Y(y)\,dy \tag{7c}$$

$$= \int_{-\infty}^{\infty} P(X \leqslant z - y|Y = y) f_Y(y)\,dy. \tag{7d}$$

Note that from line (b) to (c), we use the fact that we conditioned on $Y = y$. Applying Fact 2 (Equation 6) then gives us

$$F_Z(z) = \int_{-\infty}^{\infty} P(X \leqslant z - y) f_Y(y)\,dy. \tag{8}$$

Finally, differentiating both sides with respect to z gives

$$f_Z(z) = \frac{d}{dz} F_Z(z) \tag{9a}$$

$$= \frac{d}{dz} \int_{-\infty}^{\infty} P(X \leqslant z - y) f_Y(y)\,dy \tag{9b}$$

$$= \int_{-\infty}^{\infty} \frac{d}{dz} P(X \leqslant z - y) f_Y(y)\,dy \tag{9c}$$

$$= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\,dy. \tag{9d}$$

So

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\,dy. \tag{10}$$