Notes on Example 2.1

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I'd like to expand on Example 2.1, because the text has an interesting solution to this standard problem. Here is the mathematical problem: suppose we have two independent exponential random S and T with rates λ and μ, respectively. What is $E(S \vee T)$? (Recall that for two real numbers x and y, "$x \vee y$" means the larger of x and y.) The more pedantic approach is to write down the joint PDF

$$f_{ST}(s,t) = \lambda\mu e^{-\lambda s - \mu t} \; ; \;\; s,t \geqslant 0, \tag{1}$$

then compute the expectation

$$E(S \vee T) = \lambda\mu \int_0^\infty \int_0^\infty (s \vee t)\, e^{-\lambda s - \mu t}\, ds\, dt\;. \tag{2}$$

The integral can be split into two pieces, according to whether $s < t$ or $s > t$, and evaluated. A more elegant solution was given in the text: we know $S \wedge T$ (the minimum of S and T) is exponential with rate $\lambda + \mu$. Now, if $S < T$ (an event of probability $\lambda/(\lambda+\mu)$), then the expected time for T is again ET, since exponential random variables are memoryless. Likewise, if $T < S$, then the expected time for S is ES. Putting all this together, we have

$$E(S \vee T) = E(S \wedge T) \;\; + \;\; P(S < T) \cdot ET \;\; + \;\; P(T < S) \cdot ES \tag{3a}$$

$$= \frac{1}{\lambda+\mu} \;\; + \;\; \frac{\lambda}{\lambda+\mu} \cdot \frac{1}{\mu} \;\; + \;\; \frac{\mu}{\lambda+\mu} \cdot \frac{1}{\lambda}. \tag{3b}$$

This is a very slick argument, but you may feel it's a little too slick. The purpose of this note is to expand on it, so it doesn't look so slick.

Let's start with $E(S \vee T)$, and condition on which exponential waiting time is smaller:

$$E(S \vee T) = E(S \vee T \mid S < T) \cdot P(S < T) \quad + \quad E(S \vee T \mid T < S) \cdot P(T < S) \tag{4a}$$

$$= E(T \mid S < T) \cdot P(S < T) \quad + \quad E(S \mid T < S) \cdot P(T < S). \tag{4b}$$

Let's now compute $E(T \mid S < T)$, intentionally writing things in more abstract notation so that the key step is more apparent:

$$E(T \mid S < T) \cdot P(S < T) \tag{5a}$$

$$= \int_0^\infty \int_s^\infty t \cdot f_S(s)\, f_T(t)\, dt\; ds \tag{5b}$$

$$= \int_0^\infty f_S(s) \int_s^\infty t \cdot f_T(t \mid T > s)\, P(T > s)\, dt\; ds \tag{5c}$$

$$= \int_0^\infty f_S(s)\, P(T > s) \underbrace{\int_0^\infty t \cdot f_T(t \mid T > s)\, dt}_{(*)}\; ds \tag{5d}$$

where

$$f_T(t \mid T > s) = \begin{cases} \frac{f_T(t)}{P(T>s)}, & t > s \\ 0, & t < s \end{cases} \tag{6}$$

is the conditional PDF of T given $T > s$ and $(*)$ is the corresponding conditional expectation $E(T \mid T > s)$. Observe now that

$$f_T(t \mid T > s) = \frac{d}{dt} P(T \leqslant t \mid T > s) \tag{7a}$$

$$= \frac{d}{dt}(1 - P(T > t \mid T > s)) \tag{7b}$$

$$= \frac{d}{dt}(1 - P(T > t - s)) \tag{7c}$$

$$= \frac{d}{dt} P(T \leqslant t - s). \tag{7d}$$

The crucial step was from Eq. (7b) to (7c), where we used the memory less property of the exponential distribution. From the above, we get

$$f_T(t \mid T > s) = f_T(t - s). \tag{8}$$

(This could have obtained with a couple lines of algebra, but then it would not be so clear where we used the memoryless property!) Substituting this into (*) in Eq. (5), we get

$$E(T \mid T > s) = \int_s^\infty t \cdot f_T(t-s)\, dt \tag{9a}$$

$$= \int_0^\infty (t+s) \cdot f_T(t)\, dt \tag{9b}$$

$$= s + ET. \tag{9c}$$

(This formula can also be anticipated by the same argument used to derive Eq. (3).) So Eq. (5) becomes

$$E(T \mid S < T) \cdot P(S < T) = \int_0^\infty f_S(s)\, P(T > s)\, (ET + s)\, ds. \tag{10}$$

Finally, plugging in $f_S(s) = \lambda e^{-\lambda s}$ and $P(T > s) = e^{-\mu s}$, we get

$$E(T \mid S < T) = \lambda \int_0^\infty e^{-(\lambda+\mu)s}(1/\mu + s)\, ds \tag{11a}$$

$$= \frac{\lambda}{\lambda+\mu}\left(\frac{1}{\mu} + \frac{1}{\lambda+\mu}\right). \tag{11b}$$

Incidentally, this means

$$E(T \mid S < T) = \frac{1}{\mu} + \frac{1}{\lambda+\mu}, \tag{12}$$

which we could also have obtained using the “slick” argument.

Similarly,

$$E(S \mid T < S) = \frac{1}{\lambda} + \frac{1}{\lambda+\mu}, \tag{13}$$

so

$$E(S \vee T) = \frac{\lambda}{\lambda+\mu}\left(\frac{1}{\mu} + \frac{1}{\lambda+\mu}\right) + \frac{\mu}{\lambda+\mu}\left(\frac{1}{\lambda} + \frac{1}{\lambda+\mu}\right), \tag{14}$$

which is equivalent to Eq. (3b).