Lecture 5 notes

klin@math.arizona.edu

January 29, 2020

Today we continued Sect. 1.3, starting to prove the results covered last time:

1) (Lemma 1.10) $P_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1}$

2) (Lemma 1.3) Suppose there is $y \in S$, integer $k > 0$, and real $\alpha > 0$ such that for all $x \in S$

$$P_x(T_y \leqslant k) \geqslant \alpha. \tag{1}$$

   Then $P_x(T_y > nk) \leqslant (1-\alpha)^n$.

3) (Lemma 1.4) If $x \to y$ and $y \to z$, then $x \to z$.

4) (Theorem 1.5) If $x \to y$ and $\rho_{yx} < 1$, then x is transient.

5) (Lemma 1.6) If x is recurrent and $x \to y$, then $\rho_{yx} = 1$.

Notes

1) We have essentially proved the $k = 1$ case of Lemma 1.3 on Tuesday (see the 3-state chain example, Example 1.13); the proof for $k > 1$ is similar.

2) In the course of proving Lemma 1.10, we encountered the concept of a *stopping time*. Following our textbook, I defined a stopping time as a random variable T taking value in $\{0, 1, 2, \cdots\} \cup \{\infty\}$ such that for each n, the occurrence or non-occurrence of the event $(T = n)$ is determined by $X_0, \cdots, X_n$. For example, for the first hitting time T_x, we have

$$(T_x = n) = (X_1 \neq x) \cap \cdots \cap (X_{n-1} \neq x) \cap (X_n = x). \tag{2}$$

   In case you're wondering, a little more precisely, T is a stopping time if the event $(T = n)$ can be expressed in terms of events of the form $(X_0 = x_0), \cdots, (X_n = x_n)$ using unions, intersections, and complements. (A fully precise definition requires the language of measure theory, which is beyond the scope of this course. Those of you interested in learning more can consult, e.g., *Stochastic Processes with Applications* by Bhattacharya and Waymire or, for the adventurous, Nelson's *Radically Elementary Probability Theory*.)