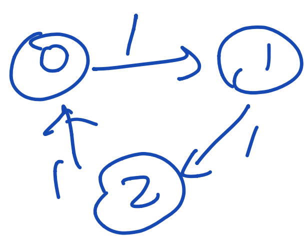
Lec 8 2/11/20

Ex. 1



Transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$P^t - \lambda I$ with $\lambda = 1$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Solve for $\pi^t = \begin{pmatrix} \pi(1) \\ \pi(2) \\ \pi(3) \end{pmatrix}$ in

$$(P^t - I)\pi^t = 0$$

Augmented matrix

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{add row 1 to 2}} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

add row 2 to 3

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} -\pi(1) + \pi(3) &= 0 \\ -\pi(2) + \pi(3) &= 0 \end{aligned}$$

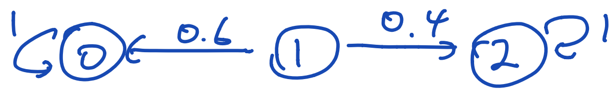
So $\pi(1) = \pi(3)$

$\pi(2) = \pi(3)$

Since $\pi(1) + \pi(2) + \pi(3) = 1$, we have

$\pi(1) = \pi(2) = \pi(3) = \frac{1}{3}$.

Ex 2. Gambler's Ruin, $N=2$



Transition Matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Transpose & subtract λI with $\lambda = 1$:

$$P^t - I = \begin{bmatrix} 1 & 0.6 & 0 \\ 0 & 0 & 0 \\ 0 & 0.4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.6 & 0 \\ 0 & -1 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

Find π^t so

$$(P^t - I)\pi^t = 0$$

Row reduction of augmented matrix

$$\begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \end{bmatrix}$$

(Multiply row 1 by $\frac{1}{0.6}$)

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (subtract $-1 \times$ row 1 from row 2 & $0.4 \times$ row 1 from row 3)

$$\Rightarrow \quad \pi(2) = 0$$

So any $\pi(1), \pi(3) \in [0,1]$ with $\pi(1) + \pi(3) = 1$ is stationary.

In particular $\pi = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\pi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent solutions.

Since the reduced row echelon form has 1 nonzero row, rank = 1 and nullity = 2, so dim(solution space) = 2, and is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

These two stationary distributions correspond to the two absorbing states.