Lecture 10 notes

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Today we discussed

- Theorem 1.20 on the asymptotic frequency of states in an irreducible recurrent chain.
- Theorem 1.21 on the stationary distribution and its relation to asymptotic frequency.

This basically follows Sects. 1.6 and 1.7 of the text.

Notes:

1) In the text's proof of Theorem 1.20, it may not be clear where recurrence is used. One place is in assuming that the chain returns to the state y infinitely often; another is assuming that $P_x(T_y < \infty)$ with probability 1 (we use assumptions (I) + (R) with Lemma 1.6).

2) The text's proof of Theorem 1.21 does not make clear that the existence of a stationary distribution implies recurrence, in the following way: suppose the chain is irreducible, and x is a state such that $\pi(x) > 0$. Then x must be recurrent.

   Proof: suppose x is transient instead; we will show $\pi(x) = 0$. First, from the definition of stationary distribution, we have

$$\sum_y \pi(y) p^k(y,x) = \pi(x) \tag{1}$$

   for all $x \in S$ and $k > 0$. Since

$$\sum_{k=1}^{n} p^k(y,x) = E_y N_n(x) \tag{2}$$

   (see proof of Lemma 1.12) we have

$$\begin{aligned}
\sum_{k=1}^{n} \sum_y \pi(y) p^k(y,x) &= \sum_y \pi(y) \sum_{k=1}^{n} p^k(y,x) && \text{(3a)}\\
&= \sum_y \pi(y) E_y N_n(x) && \text{(3b)}\\
&= n\ \pi(x). && \text{(3c)}
\end{aligned}$$

Thus

$$\frac{1}{n}\sum_{y}\pi(y)E_yN_n(x)=\pi(x). \tag{4}$$

Using Lemma 1.13 and irreducibility, one can show that if x is transient then

$$E_yN(x)=\sum_{n=1}^{\infty}p^n(y,x)<\infty. \tag{5}$$

So

$$\lim_{n\to\infty}\frac{1}{n}\sum_{y}\pi(y)E_yN_n(x)\leqslant\lim_{n\to\infty}\sum_{y}\pi(y)\frac{E_yN(x)}{n} \tag{6a}$$

$$=0 \tag{6b}$$

and $\pi(x)=0$.