Today we discussed
- The existence of stationary distributions (Theorem 1.24), with a partial proof, up to showing

\[ \sum_{z \in S} \mu_x(z)p(z, y) = \mu_x(y). \]  

(1)

Note some of the details were left out of the text and also the lecture; you will fill it in for Homework 6. (The rest of the proof is pretty clearly explained in the text.)

- The Perron-Frobenius Theorem.

**Theorem 1.24.** Here is a cleaned up proof of the following part of Theorem 1.24: suppose \((X_n)\) is an irreducible chain where every state is recurrent. For \(x, y \in S\), let

\[ \mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n). \]

(2)

We began by observing that

\[ \sum_{z} \mu_x(z) p(z, y) = \sum_{z} \sum_{n=0}^{\infty} P_x(X_n = z, T_x > n) p(z, y) \]

(3a)

\[ = \sum_{n=0}^{\infty} \sum_{z} P_x(X_n = z, T_x > n) P(X_{n+1} = y | X_n = z). \]

(3b)

If \(n = 0\), then (as explained in class)

\[ \sum_{z} P_x(X_0 = z, T_x > 0) P(X_{n+1} = y | X_n = z) = p(x, y). \]

(4)

If \(n > 0\), then \(P_x(X_n = z, T_x > n) = 0\) if \(z = x\). So we have

\[ \sum_{z} \mu_x(z)p(z, y) = p(x, y) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x > n | X_n = z) P_x(X_{n+1} = y | X_n = z) P_x(X_n = z). \]  

(5)
Recall now that for a Markov chain, the future and the past are conditionally independent given the present. (This is on Homework 6.) This and \( z \neq x \) imply

\[
P_x(T_x > n \mid X_n = z) = P_x(X_{n+1} = y \mid X_n = z) = P_x(X_n = z)
\]

because if knowing \( X_n = z \) the future and the past are conditionally independent given the present. (This is on Homework 6.) This and \( z \neq x \) imply

\[
P_x(T_x > n, X_{n+1} = y, X_n = z) = P_x(T_x > n + 1, X_{n+1} = y, X_n = z)
\]

because knowing \( X_{n+1} = y \) and \( x \neq y \), we must have \( T_x \neq n + 1 \), so that \( T_x > n + 1 \) (since \( T_x > n \) to start with). So

\[
\sum_z \mu_x(z) \ p(z, y) = p(x, y) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x > n, X_{n+1} = y, X_n = z)
\]

(7a)

\[
= p(x, y) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x > n + 1, X_{n+1} = y, X_n = z)
\]

(7b)

\[
= \sum_{n=0}^{\infty} \sum_{z} P_x(T_x > n + 1, X_{n+1} = y, X_n = z)
\]

(7c)

\[
= \sum_{n=0}^{\infty} P_x(T_x > n + 1, X_{n+1} = y).
\]

(7d)

From the second to the third line, we used similar reasoning as above (but in reverse). In the last step, we used that \( X_n \) had to be something. But this is just \( \mu_x(y) \), reindexed. So

\[
\sum_z \mu_x(z) \ p(z, y) = \mu_x(y)
\]

(8)

if \( y \neq x \).

On the other hand, if \( x = y \), then

\[
P_x(T_x > n, X_{n+1} = y, X_n = z) = P_x(T_x > n, X_{n+1} = x, X_n = z) = P_x(T_x = n + 1, X_n = z)
\]

(9a)

(9b)

because if \( X_n = z, X_{n+1} = x \), and \( x \neq z \), then we must have \( T_x = n + 1 \) (which subsumes \( T_x > n \)). So we have

\[
\sum_z \mu_x(z) \ p(z, x) = p(x, x) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x = n + 1, X_n = z)
\]

(10a)

\[
= p(x, x) + \sum_{n=1}^{\infty} P_x(T_x = n + 1)
\]

(10b)

\[
= \sum_{n=0}^{\infty} P_x(T_x = n + 1)
\]

(10c)
the last line because $P_x(T_x = 1) = p(x, x)$. By recurrence, $P_x(T_x < \infty) = 1$, so the above sums to 1. But one can check that

$$\mu_x(x) = \sum_{n=0}^{\infty} P_x(X_n = x, T_x > n)$$

(11a)

$$= P_x(X_0 = x, T_x > 0)$$

(11b)

$$= 1$$

(11c)

because $P_x(X_n = x, T_x > n) = 0$ for $n > 0$. Thus

$$\sum_z \mu_x(z) p(z, y) = \mu_x(y).$$

(12)

QED

**Perron-Frobenius Theorem.** Let $P$ be a stochastic matrix. Then

1) 1 is an eigenvalue of $P$, and it has a left eigenvector $\pi$ with all nonnegative entries.

2) $|\lambda| \leq 1$ for all eigenvalues $\lambda$ of $P$.

3) If $P$ is irreducible, then the eigenvalue 1 has multiplicity 1, i.e., there is a unique (left) eigenvector $\pi$ such that $\pi \cdot P = \pi$, $\pi(x) > 0$ for all $x$, and $\sum_x \pi(x) = 1$.

4) If $P$ is irreducible with period $d > 1$, then $P$ has exactly $d$ eigenvalues $\lambda$ with $|\lambda| = 1$. The eigenvalues all satisfy $\lambda^d = 1$, i.e., they are the $d$th roots of unity. All other eigenvalues of $P$ have absolute value $< 1$.

5) If $P$ is irreducible and aperiodic, then all eigenvalues $\lambda \neq 1$ have $|\lambda| < 1$.

6) If $P$ is irreducible and satisfies detailed balance with respect to the (unique) stationary distribution $\pi$, then all eigenvalues are real.

See [this snapshot of the Wikipedia page](https://en.wikipedia.org/wiki/Perron%E2%80%93Frobenius_theorem) for more information.