Notes for Lecture 21

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Today, we started Chapter 2 (hooray!). I mainly reviewed exponential random variables and their various properties, including:

1) If $T_1, \cdots, T_n$ are independent and $T_i \sim Exp(\lambda_i)$, then $\min(T_1, \cdots, T_n) \sim Exp(\sum_i \lambda_i)$. Moreover, if I is the index of the minimum, i.e,. $T_I = \min(T_1, \cdots, T_n)$, then I and $\min(T_1, \cdots, T_n)$ are independent.

2) If $T_1, \cdots, T_n$ are independent and $T_i \sim Exp(\lambda)$ for all i, then $\sum_i T_i$ is a $gamma(n, \lambda)$ random variable (see text).

I basically followed the proofs in the text, with the exception of the independence of I and $\min(T_1, \cdots, T_n)$. Instead of the proof in the book, I sketched how for all $i \in \{1, \cdots, n\}$ and $0 \leqslant a \leqslant b$, we have

$$P\Big((I = i) \cap (a \leqslant \min(T_1, \cdots, T_n) \leqslant b)\Big) = P(I = i) \cdot P(a \leqslant \min(T_1, \cdots, T_n) \leqslant b). \tag{1}$$

This is enough to imply that for all events $E \subset \{1, \cdots, n\}$ and $F \subset [0, \infty)$, we have $P\Big((I \in E) \cap (\min(T_1, \cdots, T_n) \in F)\Big) = P(I \in E) \cdot P(\min(T_1, \cdots, T_n) \in F)$.

Addendum, April 9, 2020. I mentioned in passing that the exponential distribution is *essentially* the only one with the memoryless property. Here is a partial "proof." (I'm not stating the underlying assumptions.) Suppose T is a continuous random variable such that $P(T > 0) = 1$ and it has the memoryless property, i.e.,

$$P(T > t + s | T > s) = P(T > t) \tag{2}$$

for all $s, t \geqslant 0$. Let

$$G(t) = P(T > t). \tag{3}$$

Then

$$\begin{aligned} G(t+s) &= P(T > t+s) \\ &= P(T > t+s, T > s) \\ &= P(T > t+s | T > s)\ P(T > s) \\ &= P(T > t)\ P(T > s) \\ &= G(t)\ G(s). \end{aligned}$$

So

$$G(t+s) - G(t) = G(t)(G(s) - 1) \tag{4}$$

Dividing both sides by s and letting $s \to 0$ yields

$$G'(t) = G(t) \cdot G'(0). \tag{5}$$

Since $G(t)$ is decreasing (or at lesat non-increasing), $G'(0) \leqslant 0$. Let $\lambda = -G'(0)$ yields $G'(t) = -\lambda G(t)$, so that $G(t) = e^{-\lambda t}$.