

# Notes for Lecture 22

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April 9, 2020

Today, we begin defining the Poisson process. I mainly followed the text, and discussed the Poisson distribution, the Poisson approximation (Theorem 2.5), the definition of the Poisson process, and the construction of Poisson processes via independent exponential waiting times.

The one deviation is proving the fact that the sum of  $n$  independent Poisson random variables is again Poisson: here I used moment generating functions, as follows: the moment generating function (MGF) of a random variable  $X$  is defined to be  $M_X(t) = E(e^{tX})$ , where  $t$  is a real variable. One property of the MGF is that if  $X$  and  $Y$  are independent, then

$$M_{X+Y}(t) = M_X(t)M_Y(t). \quad (1)$$

By induction, this means if  $X_1, \dots, X_n$  are independent, then

$$M_{X_1+\dots+X_n}(t) = M_{X_1}(t) \cdots M_{X_n}(t). \quad (2)$$

Furthermore, if  $M_X(t)$  and  $M_Y(t)$  are both defined on an interval around  $t = 0$  and  $M_X(t) = M_Y(t)$  on that interval, then  $X$  and  $Y$  have the same distribution.

Using this we can prove the result above as follows: the MGF of a Poisson random variable with rate  $\lambda$  is  $e^{\lambda(e^t-1)}$ . So if  $X_i \sim \text{Poisson}(\lambda_i)$ , then

$$\begin{aligned} M_{X_1+\dots+X_n}(t) &= M_{X_1}(t) \cdots M_{X_n}(t) \\ &= e^{\lambda_1(e^t-1)} \cdots e^{\lambda_n(e^t-1)} \\ &= e^{(\lambda_1+\dots+\lambda_n)(e^t-1)}. \end{aligned}$$

But this means  $X_1 + \dots + X_n \sim \text{Poisson}(\lambda_1 + \dots + \lambda_n)$ .

I'm not going to use MGFs heavily in this course; I only mentioned it because most of you should have seen it. Still, it's useful to know something about them. If you have not studied them before or are rusty, Chapter 8 of *Introduction to Probability* by Anderson, Seppäläinen, and Valkó has a nice discussion.

The Poisson approximation theorem (Theorem 2.5) is more important. Both our text and Anderson *et al.* cover it. Exercises 2.11–2.14 from our text are good practice.