Math 129  Elementary arithmetic, algebra, and calculus  21st century

(Originally, problems were not numbered. Problems in this list are numbered.)

1. In class, you were reminded of the simple definition of the absolute value function.

Give it here by filling in the blanks below.

Keep it simple; don’t use square roots or quotients, just simple algebra formulas.

**SOLUTION.**

\[
|x| = x \quad \text{if } x > 0; \\
|x| = 0 \quad \text{if } x = 0; \\
|x| = -x \quad \text{if } x < 0.
\]

[This is exactly the formula which was given in class, and similar to formula in Sect. 2.6 of textbook.]

2. To emphasize that the absolute value function is a function, we will write \( \text{abs}(x) = |x| \).

Rewrite the three equalities above using \( \text{abs}(x) \) instead of \( |x| \).

**SOLUTION.** Since \( \text{abs}(x) = |x| \), we just copy what we have above (why do anything else?).

\[
\text{abs}(x) = x \quad \text{if } x > 0; \\
\text{abs}(x) = 0 \quad \text{if } x = 0; \\
\text{abs}(x) = -x \quad \text{if } x < 0.
\]

3. Draw a graph of the absolute value function for approximately \(-3 < x < 3\).

**SOLUTION** not given here, but everyone should know what the graph of the absolute value function looks like; it’s V-shaped with a vertex at the origin.

4. Now give the appropriate formulas for the derivative, \( \text{abs}'(x) = \frac{d}{dx} |x| \).

Get your answers either from the algebraic formulas for absolute value given above, or from the graph. Some answers may be easier using the formulas, some may be easier using the graph.

[Say what the derivative is, simply, using = notation appropriately]

**SOLUTION.** [As stated in Problem 1, keep it simple. As stated here, use the algebraic formulas or the graph.]

If \( x > 0 \), \( \text{abs}'(x) = 1 \), because \( \text{abs}(x) = x \) if \( x > 0 \) [so the derivative is 1];

If \( x = 0 \), \( \text{abs}'(x) \) is undefined, because the graph has a corner when \( x = 0 \);

If \( x < 0 \), \( \text{abs}'(x) = -1 \), because \( \text{abs}(x) = -x \) if \( x < 0 \) [so the derivative is -1].
The purpose of these problems (as is the case with many problems on written homework) is to UNDERSTAND, in this case to understand the absolute value function, a function which everyone has seen before but many do not understand, and use that understanding to obtain a formula for the derivative.

That is why, in the first problem, it is stated “keep it simple; don't use square roots or quotients”.

The formulas given above, and the discussion above (and below) are SIMPLE. Any student who understands that “derivative” means “slope” can understand, from the graph of the absolute value function, that the derivative has the values “-1, does not exist, and 1”. Similarly, anyone who can do the most basic derivatives formulas can understand, from the formula given for the absolute value, that the derivative has these values.

On the other hand, some students wrote down a completely different formula for the derivative, something like $x/|x|$, without any attempt at explanation (ONE student did derive/explain the formula based on a different definition of absolute value).
5. Let \( f(x) = \text{abs}(x-3) \). Use the chain rule and the results above to find the derivative \( f'(x) \).

Of course, there will be three cases, just as above when we wrote the formulas for \( \text{abs}'(x) \).

**SOLUTION.** Here is the basic idea (the chain rule approach will be discussed below); we can just use the result in #4. Or just look at the graph of \( |x-3| \).

If \( x - 3 > 0 \), \( \text{abs}'(x-3) = 1 \), so if \( x > 3 \), \( \text{abs}'(x-3) = 1 \);

If \( x - 3 = 0 \), \( \text{abs}'(x-3) \) is undefined, so if \( x = 3 = 0 \), \( \text{abs}'(x-3) \) is undefined;

If \( x - 3 < 0 \), \( \text{abs}'(x-3) = -1 \), so if \( x < 3 \), \( \text{abs}'(x-3) = -1 \).

**CHAIN RULE:** If \( f(x) = g(h(x)) \), then \( f'(x) = g'(h(x)) h'(x) \).

So, if \( f(x) = \text{abs}(h(x)) \), then \( f'(x) = \text{abs}'(h(x)) h'(x) \).

So (again), if \( f(x) = \text{abs}(x-3) \), then \( f'(x) = \text{abs}'(x-3) \frac{d}{dx}(x-3) \).

Since everyone knows that \( \frac{d}{dx}(x-3) = 1 \), we get \( f'(x) = \text{abs}'(x-3) \). See solution above.

**Very Important Point** - The “critical point” here (both in the sense of English “critical” [“important”] and in the sense of mathematical “critical point”) is when \( x = 3 \), NOT when \( x = 0 \).

Final answer (after explanation):

If \( x > 3 \), \( f'(x) = \text{abs}'(x-3) = 1 \);

If \( x = 3 \), \( f'(x) \) and \( \text{abs}'(x-3) \) are undefined.

If \( x < 3 \), \( f'(x) = \text{abs}'(x-3) = -1 \).